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Distributed $(\Delta + 1)$-coloring in the physical model

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ABSTRACT

In multi-hop radio networks, such as wireless ad-hoc networks and wireless sensor networks, nodes employ a MAC (Medium Access Control) protocol such as TDMA to coordinate accesses to the shared medium and to avoid interference of close-by transmissions. These protocols can be implemented using standard node coloring. The $(\Delta+1)$-coloring problem is to color all nodes in as few timeslots as possible using at most $\Delta+1$ colors such that any two nodes within distance R are assigned different colors, where R is a given parameter and Δ is the maximum degree of the modeled unit disk graph using R as a scaling factor. Being one of the most fundamental problems in distributed computing, this problem is well studied and there is a long chain of algorithms prescribed for it. However, all previous works are based on abstract models, such as message passing models and graph based interference models, which limit the utility of these algorithms in practice. In this paper, for the first time, we consider the distributed $(\Delta + 1)$-coloring problem under the more practical SINR interference model. In particular, without requiring any knowledge about the neighborhood, we propose a novel randomized $(\Delta + 1)$-coloring algorithm with time complexity $O(\Delta \log n + \log^2 n)$. For the case where nodes cannot adjust their transmission power, we give an $O(\Delta \log^2 n)$ randomized algorithm, which only incurs a logarithmic multiplicative factor overhead.

1. Introduction

The node coloring problem underpins the design of interference avoidance mechanisms in many multi-hop radio networks including wireless ad-hoc networks and wireless sensor networks. In these networks, radio communications are subject to interference, and messages may be lost due to interference. Without any interference avoidance mechanism, coordinating the nodes to achieve efficient and reliable communication can be quite a complex task. Traditionally, nodes employ MAC (Medium Access Control) protocols to coordinate their accesses to the shared medium and to avoid interference of close-by transmissions, such as TDMA (Time Division Multiple-Access). These MAC protocols can all be reduced to the classical node coloring problem. For example, by assigning different colors to different time slots in a TDMA scheme, a proper coloring with parameter d corresponds to a MAC layer without "close-by" interference, i.e., no two nodes within distance d of each other transmit at the same time. In [4], it is shown that even under the more elaborate (but also more realistic) SINR model, we can still implement an interference free TDMA-like MAC protocol by computing a proper coloring for a well defined d if we adopt a uniform power assignment (all the nodes employ the same transmission power). Conventionally, the node coloring problem is one of the most fundamental problems of symmetry breaking, and therefore has attracted a great deal of attention in the distributed computing community.

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Almost all previous works to derive distributed node coloring algorithms assume the graph based model in which interference is represented by a localized function—a message can be correctly received only if there are no other simultaneously transmitting senders in the receiver's neighborhood. However, in multi-hop radio networks, interference is cumulative and is contributed to by all simultaneously transmitting nodes, near by and far away. The physically based Signal-to-Interference-plus-Noise-Ratio (SINR) model [6] captures this reality in wireless networks more faithfully. Under the SINR model, the signal strength fades with distance to the power of some path-loss exponent α and a message can be successfully received if the ratio of the received signal strength and the sum of the interference caused by simultaneously transmitting nodes plus noise is above a certain hardware-defined threshold β.

1.1. Related work

In the absence of global knowledge, to derive a $(\Delta+1)$-coloring in a distributed manner is challenging and has attracted much attention in the distributed computing community for more than two decades. The traditional message passing model was considered in the beginning. Following Cole and Vishkin who presented the first distributed $(\Delta+1)$-coloring for rings in [2], a long line of papers were devoted to this problem. The state-of-the-art results are the $O(\Delta)+\frac{1}{2}\log^* n$ algorithm for arbitrary graphs in [1] and the optimal $O(\log^* n)$ algorithm for bounded-independence graphs in [18]. However, the message passing model abstracts away some crucial elements of wireless networks, including interference, collision and asynchrony. Taking interference into account and assuming a locally synchronous circumstance, Schneider and Wattenhofer [19] proposed a distributed $(\Delta+1)$-coloring algorithm with running time $O(\Delta+\log\Delta\log n)$ and $O(\Delta+\log^2 n)$ with and without knowledge of Δ respectively. When further considering asynchrony, and assuming prior knowledge of n and Δ, Moscibroda and Wattenhofer [14] gave an $O(\Delta\log n)$ distributed coloring algorithm for the simple unit disk graph model which only considers direct interferences from neighbors. In an extended version [15], the result was generalized for the bounded-independence graph. In a recent paper [4], Derbel and Talbi showed that the algorithm in [15] also works under the SINR model within the same time bound. However, all the above three algorithms need $O(\Delta)$ colors instead of at most $\Delta+1$ colors.

In the SINR model, the interference is modeled as a global function, which greatly adds to the difficulty of designing efficient distributed algorithms with global performance guarantee. In spite of the difficulty, there are still many recent works facing up to the challenge and designing efficient distributed algorithms for various fundamental problems in wireless networks, e.g., the dominating set problem [17], the contention resolution problem [13,8,7], the broadcast problem [3,11,12,10,21,22] and the local broadcasting problem [5,9,23,20].

1.2. Our contribution

To the best of our knowledge, this work is the first one that considers the distributed $(\Delta+1)$-coloring problem under the physical model. Without relying on any knowledge about the neighborhood, we give an $O(\Delta\log n+\log^2 n)$ time randomized distributed $(\Delta+1)$-coloring algorithm for asynchronous wake-up multi-hop radio networks under the physical model. Our result even matches the coloring algorithm in [4] for large Δ, e.g., $\Delta\geq\log n$, which needs a linear estimate of Δ and uses $O(\Delta)$ colors. In our algorithm, we adopt a clustering based coloring strategy, i.e., a Maximal Independent Set (MIS) is first computed, and then the nodes in the MIS assign colors to their neighbors. To be able to employ the strategy, we first show that the MIS algorithm in [16] still works under the SINR model by carefully tuning the parameters. This algorithm is of independent interest, since it is the first MIS algorithm in the physical model.

Furthermore, assuming nodes cannot adjust their transmission powers, we then give a distributed $(\Delta+1)$-coloring algorithm with time complexity $O(\Delta\log^2 n)$ by iteratively carrying out the MIS algorithm, which also does not need any knowledge about the neighborhood.

1.3. Organization

The remaining part of this paper is organized as follows. In Section 2, we give a formal introduction of the network model and the problem. We present the $O(\Delta\log n+\log^2 n)$ time $(\Delta+1)$-coloring algorithm and the $(\Delta+1)$-coloring algorithm under uniform power assignment in Section 3 and Section 4, respectively. Section 5 concludes the paper.

2. Problem definition and network model

The network consists of n nodes that are placed arbitrarily in a Euclidean space. Each node v has a unique ID_v. Nodes cannot sense the physical carrier. The only prior knowledge given to the nodes is a polynomial estimate of the number n of nodes in the network, but they are clueless about the number of nodes in its close proximity.

We assume that the time is divided into timeslots by which the nodes are synchronized. In each timeslot, a node can either transmit or listen, but cannot do both. Nodes may wake up asynchronously or by an incoming message and not according to any global clock. Each node executes the algorithm based on its own clock. Note that the assumption of synchronous timeslots is just for ease of analysis; our algorithm does not rely on any synchrony in any way.

Message transmissions in wireless networks are subject to interference. We adopt the physical interference model (the SINR model) [6] which is a close approximation to the physical reality. Denote by $d(x,y)$ the Euclidean distance between two nodes x and y. In the SINR model, a message sent by node u to node v can be correctly received at v iff

$$\frac{\frac{P_u}{d(u,v)^\alpha}}{N + \sum_{w \in V \setminus \{u,v\}} \frac{P_w}{d(w,v)^\alpha}} \geq \beta, \tag{1}$$

where P_u (P_w) is the transmission power for node u (w); α is the path-loss exponent whose value is normally between 2 and 6; β is a hardware-determined threshold value which is greater than 1; N is the ambient noise, and $\sum_{w \in V \setminus \{u,v\}} \frac{P_w}{d(w,v)^\alpha}$ is the interference experienced by the receiver v caused by all simultaneously transmitting nodes in the network.

The transmission range R_T of a node v can be defined as the maximum distance at which a node u can receive a clear transmission from v ($SINR \geq \beta$) when there are no other simultaneous transmissions in the network. From the SINR condition (1), $R_T \leq R_{max} = (\frac{P}{\beta \cdot N})^{1/\alpha}$ for the given power level P. We further assume that $R_T < R_{max}$ and define $R_T = (P/cN\beta)^{1/\alpha}$, where $c > 1$ is a constant determined by the environment.

In this work, we consider two application scenarios. In the first scenario, nodes can adjust their transmission powers up to a constant factor. In the second scenario, nodes use the same transmission power and cannot adjust the power during the algorithm execution. The reason that we do not take "arbitrary" power control into consideration is that in reality, arbitrary power control is very hard to implement given the wide variations in the relevant chipsets.

Because nodes may be able to adjust their transmission powers, and different power assignments correspond to different communication graphs. So here we do not define the node coloring problem on the communication graph as in previous works under the graph based model. Instead, we define the problem in terms of a given distance parameter R. In particular, given a distance parameter R, we say two nodes u and v are neighbors if $d(u,v) \leq R$. The neighborhood of a node v is the set of all its neighbors, denoted by $N(v)$. Additionally, we use $N[v]$ to denote the set $N(v) \cup \{v\}$. For a node v, we denote by Δ_v the number of nodes in v's neighborhood. We write $\Delta = \max_{v \in V} \Delta_v$. A set of nodes S is called an independent set if any two nodes of S are not in each other's neighborhood. An independent set S is maximal if for any node v, either $v \in S$ or there is a node $u \in N(v)$ such that $u \in S$. A node coloring is proper if each set of nodes with the same color is an independent set, i.e., the distance between any two nodes with the same color is larger than R. The $(\Delta + 1)$-*coloring problem* then is to color all nodes properly in as few timeslots as possible using at most $\Delta + 1$ colors.

We define a node v's running time as the interval from the timeslot when v starts executing the algorithm to the timeslot when v quits the algorithm. The time complexity of the algorithm is the maximum of all the nodes' running times.

In subsequent sections, when we say "an event occurs with high probability" we mean that the event occurs with probability at least $1 - n^{-c}$ for a constant $c > 0$, and "a node correctly gets a color" means that the resulting coloring of the network is proper. Greek letters represent constants. Definition 1 and Lemma 1 in the following will be useful in the analysis of the algorithms.

Definition 1. For a node $v \in V$, the probabilistic interference at v, Ψ_v, is defined as the expected interference experienced by v in a certain timeslot t.

$$\Psi_v = \sum_{u \in V \setminus \{v\}} \frac{P_u p_u}{d(u,v)^\alpha}, \tag{2}$$

where P_u is the transmission power and p_u is the sending probability of node u in timeslot t.

Lemma 1. *(See [5].) Consider two disks D_1 and D_2 of radii R_1 and R_2, $R_1 > R_2$, we define $\chi(R_1, R_2)$ to be the smallest number of disks D_2 needed to cover the larger disk D_1. It holds that: $\chi(R_1, R_2) \leq \frac{2\pi}{3\sqrt{3}} \cdot \frac{(R_1+2R_2)^2}{R_2^2}$.*

3. An $O(\Delta \log n + \log^2 n)$ $(\Delta + 1)$-coloring algorithm

In this section, we introduce the distributed randomized coloring algorithm, which is shown as Algorithm 1. It is assumed that nodes can adjust the transmission power up to a constant factor. Every node v possesses a color list from which it chooses a color. Without loss of generality, we assume that all nodes' color lists are $\{0, 1, \ldots, n-1\}$, where n is the estimate of the number of nodes.

The main idea of the algorithm is as follows. A set of leaders is first elected. These leaders will coordinate the color choosing processes of their neighbors that are within distance R. Specifically, each non-leader sends a message to report its existence to its leader. After receiving these messages from the non-leaders, the leader assigns each non-leader a non-overlapping interval of timeslots for choosing its color and informing its neighboring nodes of the choice, which guarantees that any neighboring non-leaders dominated by the same leader will not choose the same color. In the coloring algorithm, the leader is elected by executing a Maximal Independent Set (MIS) algorithm. In particular, we show that the distributed MIS algorithm in [16] still works under the SINR model by carefully tuning the parameters. Due to asynchrony, when some nodes execute the MIS algorithm, other nodes may be carrying out other operations in the coloring algorithm. Here we show that under such an asynchronous circumstance, the MIS algorithm can still correctly output an independent set in any timeslot with high probability. The MIS algorithm is shown as Algorithm 2. Furthermore, the leader is elected in terms of $3R$, i.e., any pair of leaders has distance larger than $3R$. The purpose of doing this is to make sure that any two neighboring non-leaders that are dominated by different leaders will not choose their colors at the same time. Combining with the fact that any two neighboring non-leaders dominated by the same leader would not choose their colors simultaneously, it is guaranteed that any pair of neighboring nodes will not choose the same color. Thus the MIS algorithm is executed in terms of $3R$. After satisfying a specified condition, a leader will quit the algorithm such that nodes within distance $3R$ that have not chosen their colors can start their coloring processes. The quit condition can guarantee that with high probability, when a leader quits the algorithm, all non-leaders dominated by it have chosen their colors.

In Algorithm 1, there are four states: nodes in state $\mathcal{G}$ are leaders; nodes in state $\mathcal{C}_1$ are non-leaders that are competing for the right to choose a color; nodes in state $\mathcal{C}_2$ are non-leaders that have chosen a color and are informing their neighbors of their choice; state $\mathcal{S}$ is for making sure that newly waken-up nodes will not disturb the coloring processes of nearby nodes. We also use several control messages to guarantee the correctness and efficiency of the coloring algorithm. The *DoNotTransmit* message is used by a leader to ask nodes within $3R$ to keep silence; nodes within distance $3R$ from the leader that received this message will join state $\mathcal{S}$. The *StartColoring* message is used by a leader to inform its neighbors within R to join its cluster and start the coloring process. All its neighbors within distance R received this message will join state $\mathcal{C}_1$. The *RequestColor* message is for a non-leader in state $\mathcal{C}_1$ to inform its leader of its existence. The *Grant* message is used by a leader to prompt a non-leader in its cluster to choose a color; after receiving this message, the non-leader in question joins state $\mathcal{C}_2$. The *Grant* message also works as a control message to adjust non-leaders' transmission probabilities. The *Color* message is used by a non-leader in state $\mathcal{C}_2$ to inform its neighbors of its choice. The *StartTransmit* message is used by a leader to remove the restriction on nodes within distance $3R$ caused by the *DoNotTransmit* message transmitted by it before. Furthermore, in Algorithm 1, we assign each leader a set Q to store the IDs of non-leaders that have sent a *RequestColor* message to the leader. We use the set T for a non-leader to store the colors that have been chosen by its neighbors.

In the coloring algorithm, nodes adopt different transmission powers when executing different operations. Generally speaking, nodes adopt the transmission power of $P_M = c \cdot 3^\alpha N\beta R^\alpha$ when they execute the MIS algorithm and transmit a *StartTransmit* message in state $\mathcal{G}$, while nodes adopt the transmission power of $P_C = cN\beta R^\alpha$ when they perform other operations. By the definition in Section 2, the transmission ranges of nodes are $3R$ and R for P_M and P_C, respectively. Next we discuss the algorithm in more details.

3.1. Description of Algorithm 1

After waking up, a node v will first wait for at most $2(\mu_1+\mu_2)\log n$ timeslots (both μ_1 and μ_2 are constants which will be defined later). During the process, if v received a $DoNotTransmit_u$ message, it enters state $\mathcal{S}$ and adds u to its forbidden set F_v. Otherwise, it starts executing the MIS algorithm after waiting for $2(\mu_1+\mu_2)\log n$ timeslots. After executing the MIS algorithm (Algorithm 2), each node will either join state $\mathcal{M}$ meaning that it is a member of the computed independent set, or join state $\mathcal{S}$. Here we must point out a difference of our MIS algorithm from that in [16] regarding state $\mathcal{M}$. In our algorithm, when a node v joins state $\mathcal{M}$, it first uses $\mu_2 \log n$ timeslots to wake up all nodes within distance $3R$ by transmitting a message with constant probability. Then v transmits a $DoNotTransmit_v$ message forcing all nodes within distance $3R$ to join state $\mathcal{S}$. Having done this, v joins state $\mathcal{G}$.

In the coloring algorithm, the leaders in state $\mathcal{G}$ first choose color 0 as its own color. Then they transmit a *StartColoring* message bidding their neighbors within distance R to join state $\mathcal{C}_1$. While in state $\mathcal{G}$, a node v adds each of its neighbors that send a *RequestColor* message to v to a set Q_v. If Q_v is not empty, it deletes the first node u from Q_v and transmits a $Grant_u$ message with constant probability for $2\mu_1 \log n$ timeslots, which ensures that the *Grant* message can be received by all neighbors with high probability. As described later, this $Grant_u$ message has two functions: first, after receiving the message, u will start choosing its color; second, neighbors of v will adjust the transmission probability based on the reception of *Grant* messages. We assign two counters c_v and b_v to each node v in state $\mathcal{G}$. c_v is used to count the number of timeslots that v has not received any *RequestColor* message since the last one, while b_v is for counting the number of *Grant* messages that have been transmitted by v. These two counters are set in order to guarantee that with high probability, v will not quit the algorithm until all neighbors have been colored. Then if Q_v is empty and $c_v > b_v \cdot 5\mu_1 \log n + 3\mu_1 \log^2 n + \mu_1 \log n$, v quits the algorithm after transmitting a $StartTransmit_v$ message for $\mu_2 \log n$ timeslots adopting power P_M. By doing so, v removes its restriction on nodes within distance $3R$ caused by the message $DoNotTransmit_v$.

For each node u in state $\mathcal{S}$, it will do nothing except listening. When u stays in state $\mathcal{S}$, it adds the nodes that send *DoNotTransmit* messages to u into its forbidden set F_u, and it removes a node v from F_u if it receives a message $StartTransmit_v$. Node u will not leave state $\mathcal{S}$ until F_u is empty or it receives a *StartColoring* message from a leader node v. For the first case, u starts executing the MIS algorithm. For the second case, it joins state $\mathcal{C}_1$ and starts competing for the right to choose a color. After joining state $\mathcal{C}_1$, node u starts transmitting a $RequestColor_u$ message with a small initial transmission probability. Then if u did not receive any *Grant* message and did not change its transmission probability for $3\mu_1 \log n$ timeslots, it doubles the transmission probability. While in state $\mathcal{C}_1$, if u receives a *Grant* message and the *Grant* message is not for u, it halves the transmission probability. If the received *Grant* message is for u, u would join state $\mathcal{C}_2$. The transmission probability adjustment strategy guarantees that on the one hand, the sum of transmission probabilities in any local region of the network can be bounded with high probability, which helps bound the interference caused by simultaneously transmitting nodes; and on the other hand, each node u in $\mathcal{C}_1$ can send its $RequestColor_u$ message to the leader with high probability in $O(\Delta \log n + \log^2 n)$ timeslots, such that the node will finally get the $Grant_u$ message from

Algorithm 1 $(\Delta+1)$-Coloring.

Initially, $p_v = \frac{2^{-\omega-1}}{n}$; $c_v = 0$; $b_v = 0$; $t_v = 0$; $Q_v = \emptyset$; $T_v = \emptyset$; $\omega = 6.4$;
Upon node v **wakes up**
wait for $2(\mu_1 + \mu_2)\log n$ timeslots
if Received *DoNotTransmit*$_u$ from node u **then** add u into F_v; *state* $= \mathcal{S}$;
Else execute the MIS algorithm adopting transmission power P_M **end if**
Message Received
if Received *Color*$_w$ **then** delete the color in *Color*$_w$ from its color list **end if**
Node v **in state** $\mathcal{G}$
choose color 0;
for $\mu_1 \log n$ timeslots **do** transmit *StartColoring*$_v$ adopting power P_C with probability $2^{-\omega}$ **end for**
if Q_v is not empty **then**
    $b_v = b_v + 1$;
    for $2\mu_1 \log n$ timeslots **do** delete the first node u from Q_v and transmit *Grant*$_u$ adopting power P_C with probability $2^{-\omega}$; $c_v = c_v + 1$ **end for**
else $c_v = c_v + 1$ **end if**
if Q_v is empty and $c_v > b_v \cdot 5\mu_1 \log n + 3\mu_1 \log^2 n + \mu_1 \log n$ **then**
    for $\mu_2 \log n$ timeslots **do** transmit *StartTransmit*$_v$ adopting power P_M with probability $2^{-\omega}$ **end for**
quit
end if
Message Received
if Received *RequestColor*$_u$ **then** add u into Q_v; $c_v = 0$ **end if**
Node v **in state** $\mathcal{S}$
if F_v is empty **then** execute the MIS algorithm with power P_M **else** listen **end if**
Message Received
if Received *DoNotTransmit*$_w$ from node w **then** add w into F_v **end if**
if Received *Color*$_w$ **then** delete the color in *Color*$_w$ from its color list **end if**
if Received *StartTransmit*$_w$ from node w **then** delete w from F_v **end if**
if Received *StartColoring*$_w$ from node w **then** *state* $= \mathcal{C}_1$ **end if**
Node v **in state** $\mathcal{C}_1$
$t_v = t_v + 1$
if $t_v > 3\mu_1 \log n$ **then** $p_v = 2p_v$; $t_v = 0$ **end if**
transmit *RequestColor*$_v$ adopting power P_C with probability p_v;
Message Received
if received *Grant*$_v$ **then** *state* $= \mathcal{C}_2$ **end if**
if received *Grant*$_w$ for some node w that has not been received before **then** $p_v = p_v/2$; $t_v = 0$ **end if**
if Received *Color*$_w$ **then** delete the color in *Color*$_w$ from its color list **end if**
Node v **in state** $\mathcal{C}_2$
choose the first available color from its color list;
for $\mu_1 \log n$ timeslots **do** transmit a message *Color*$_v$ containing its color adopting power P_C with probability $2^{-\omega}$ **end for**
quit;

the leader. After joining $\mathcal{C}_2$, u chooses the first color remaining in its color list except color 0 and transmits a *Color*$_u$ message with constant probability for $\mu_1 \log n$ timeslots. This ensures that it can inform its neighbors of its choice with high probability. After waking up, each node will delete the color in the received *Color* message from its color list; hence it will not choose a color that has been chosen by its neighbors.

In order to make sure Algorithm 1 is correct with high probability, we assign $\mu_1 = 2^{\omega+7} \cdot 4^{3\cdot 2^{1-\omega}\cdot\chi(R_I+R,0.5R)}/(1-1/\rho)$, where ρ and R_I (Eq. (3) below) are constants defined in the following analysis. The values of ω and μ_2 are determined by the MIS algorithm, which can be found in the analysis of the MIS algorithm in Appendix A.

3.2. Description of the MIS algorithm

The MIS algorithm is given as Algorithm 2, which is the same as that in [16] except for the operations in the last state $\mathcal{M}$. The basic idea of the algorithm is that through competition in two stages, $\mathcal{A}$ and $\mathcal{B}$, the number of competitors is reduced until there is only one active node left in the required range. Specifically, after the first competition stage ($\mathcal{A}$), the number of neighboring nodes that will participate in the second stage is at most $O(\log n)$. In the second stage, candidates use a counter to record the time passed since their first transmission or the last reception of a sufficiently close neighbor's counter. When a node's counter exceeds a specified threshold, the node joins the MIS and makes all its neighbors stop the competition via a control message. For further details, please refer to [16]. In order to compute an MIS in terms of R_{mis}, the transmission power of the nodes is assigned as $P_{mis} = cN\beta R_{mis}^{\alpha}$. In the coloring algorithm, R_{mis} should be $3R$. Based on a sufficient condition for successful transmissions under the SINR model, we can show that, as long as the sum of transmission probabilities of nodes in any local region executing other algorithms can be bounded by a constant in any timeslot, each node can correctly decide whether to join the MIS or state $\mathcal{S}$ in $O(\log^2 n)$ timeslots. The analysis is similar to that in [16], which we include in Appendix A for the sake of completeness. Furthermore, the values of the constant parameters used in Algorithm 2 are given in the analysis of the MIS algorithm in Appendix A.

Algorithm 2 MIS.

Upon node v begins to execute the algorithm:

$step = count = 0$; $state = \mathcal{W}$; $q_v = \frac{2^{-\omega-1}}{n}$; $q_B = \frac{\tau}{2^{\omega}\log n}$; $q_C = 2^{-\omega}$; $P_{mis} = cN\beta R_{mis}^{\alpha}$

Node v in state $\mathcal{W}$

while $state = \mathcal{W}$ **do**
  $step := step + 1$;
  if $step \geq 4\theta\delta\log^2 n + 2\mu_2\log n$ **then** $state := \mathcal{A}$; $step := 0$ **end if**
Message Received
  if m_A received **then** $step := 0$ **end if**
  if $DoNotTransmit_u$ received **then** add u into F_v and $state = \mathcal{S}$ **end if**

node v in state $\mathcal{A}$

while $state := \mathcal{A}$ **do**
  $step := step + 1$
  if $step \geq \lambda\log n$ **then** $q_v = 2q_v$; $step := 0$ **end if**
  $s := \begin{cases} 1, & \text{with probability } q_v \\ 0, & \text{with probability } 1 - q_v \end{cases}$
  if $s = 1$ **then** transmit m_A with power P_{mis}; $state := \mathcal{B}$; $step := 0$ **end if**
Message Received
  if m_A received **then** $state := \mathcal{W}$; $step := 0$ **end if**
  if $DoNotTransmit_u$ received **then** add u into F_v and $state = \mathcal{S}$ **end if**

node v in state $\mathcal{B}$

while $state = \mathcal{B}$ **do**
  $step := setp + 1$
  if $step > 2\mu_2\log n$ **then** $count := count + 1$; transmit $m_B(count)$ with probability q_B and transmission power P_{mis} **end if**
  if $count \geq \delta\log^2 n$ **then** $state := \mathcal{M}$ **end if**
Message Received
  if $m_B(count')$ received **then**
    if $|count - count'| \leq 2\mu_2\log n$ **then** $count := 0$; $step := 0$ **end if**
  end if
  if $DoNotTransmit_u$ received **then** add u into F_v and $state = \mathcal{S}$ **end if**

node v in state $\mathcal{M}$

  for $\mu_2\log n$ time-slots **do** transmit a waking-up message with probability q_C and transmission power P_{mis} **end for**
  for $\mu_2\log n$ time-slots **do** transmit a message $DoNotTransmit_v$ with probability q_C and transmission power P_{mis} **end for**
  $state = \mathcal{G}$;

3.3. Analysis

In this section, we show that with high probability, each node can correctly get a color after executing Algorithm 1 for $O(\Delta\log n + \log^2 n)$ timeslots, and the total number of colors used is at most $\Delta + 1$. We first give some definitions and notations that will be used in the subsequent analysis. A new parameter R_I is defined as follows, for bounding the interference.

$$R_I = R\left(2^{7-\omega}3^{\alpha+1}\sqrt{3}\pi\rho\beta \cdot \frac{1}{1-1/c} \cdot \frac{\alpha-1}{\alpha-2}\right)^{1/(\alpha-2)}, \tag{3}$$

where ρ is a constant larger than 1. We choose ρ such that $R_I > 2R$. Furthermore, we denote T_i, D_i and I_i as the disks centered at node i with radii R, $\frac{R}{2}$ and R_I, respectively. By E_i^r we denote the disk centered at node i with radius r. We also use T_i, D_i, I_i and E_i^r to denote the set of nodes in T_i, D_i, I_i and E_i^r, respectively.

Before analyzing Algorithm 1, we first give a lemma on the time complexity and the correctness of the MIS algorithm, the proof of which is in Appendix A.

Lemma 2. *With probability $1 - O(n^{-3})$, every node $v \in V$ decides whether it should join the computed independent set or state $\mathcal{S}$ after executing the MIS algorithm for at most $O(\log^2 n)$ timeslots. Furthermore, with probability at least $1 - O(n^{-3})$, in any timeslot t, the independent set computed by the MIS algorithm is correct.*

The following property is also proved to be correct with probability at least $1 - O(n^{-3})$ in the analysis of the MIS algorithm in Appendix A.

Property 1. *For any disk D_i and in any timeslot t throughout the execution of the algorithm, the sum of transmission probabilities of nodes that are executing the MIS algorithm is at most $3 \cdot 2^{-\omega}$.*

In order to bound the interference, we present Property 2 which can be proved to hold with probability at least $1 - O(n^{-1})$ by Lemma 8.

Property 2. *For any disk D_i and in any timeslot t throughout the execution of the algorithm,*

(i) there is at most one node in state $\mathcal{C}_2$;
(ii) the sum of transmission probabilities of nodes in state $\mathcal{C}_1$ is at most $\sum_{u\in\mathcal{C}_1} \le 2^{-\omega}$;
(iii) there is at most one node in state $\mathcal{G}$.

Based on Properties 1, 2 and the transmission probability in each state, we can bound the sum of transmission probabilities as follows.

Lemma 3. *Assume that Property 1 and Property 2 hold. For any disk D_i and in any timeslot t throughout the execution of the algorithm, the sum of transmission probabilities can be bounded as $\sum_{v\in D_i} p_v \le 3\cdot 2^{1-\omega}$.*

In the following Lemma 4, we give a sufficient condition for a successful transmission.

Lemma 4. *Assume that Property 1 and Property 2 hold. If node v is the only sending node in $E_v^{R_I+R}$, with probability $1-\frac{1}{\rho}$, the message sent by v will be received successfully by all nodes in T_v.*

Proof. We first bound the interference at a receiver $u\in T_v$ caused by nodes outside I_u.

Claim. For a node $u\in T_v$, the probabilistic interference caused by nodes outside I_u can be bounded as: $\Psi_u^{w\notin I_u} \le \frac{(1-1/c)P_C}{\rho\beta R^\alpha}$.

Proof. By Lemma 1 and Lemma 3, the sum of transmission probabilities in each T_i can be bounded as follows:

$$\sum_{w\in T_i} p_w \le \frac{2\pi}{3\sqrt{3}}\cdot\frac{(R+2\cdot\frac{R}{2})^2}{(\frac{R}{2})^2}\cdot\sum_{w\in D_j} p_v \le \frac{64\pi}{\sqrt{3}\cdot 2^\omega}. \tag{4}$$

Let $R_l=\{w\in V : lR_I \le d(u,w) \le (l+1)R_I\}$ and $\mathcal{I}$ be a maximum independent set in R_l. Clearly, $\mathcal{I}$ is also a dominating set in R_l. Thus $\sum_{i\in\mathcal{I}} T_i$ covers all nodes in R_l. Furthermore, all disks D_i for every $i\in\mathcal{I}$ are mutually disjoint because of the independence of $\mathcal{I}$. Note that all these disks are located inside the extended region $R_l^+=\{w\in V : lR_I-\frac{R}{2}\le d(u,w)\le (l+1)R_I+\frac{R}{2}\}$. Thus $|\mathcal{I}|\le Area(R_l^+)/Area(disk(R/2))$. Then the probabilistic interference caused by nodes inside R_l is bounded as follows:

$$\begin{aligned}
\Psi_u^{R_l} &= \sum_{w\in R_l}\Psi_u^w \le \frac{Area(R_l^+)}{Area(disk(R/2))}\cdot\max_{i\in I}\left\{\sum_{w\in T_i\cap R_l}\frac{P_M\cdot p_w}{(lR_I)^\alpha}\right\}\\
&\le \frac{Area(R_l^+)}{Area(disk(R/2))}\cdot\frac{64\pi}{\sqrt{3}\cdot 2^\omega}\cdot\frac{P_M}{(lR_I)^\alpha}\\
&= \frac{\pi(((l+1)R_I+R/2)^2-(lR_I-R/2)^2)}{\pi(R/2)^2}\cdot\frac{64\pi}{\sqrt{3}\cdot 2^\omega}\cdot\frac{P_M}{(lR_I)^\alpha}\\
&= \frac{4(2l+1)(R_I^2+R_IR)}{R^2}\cdot\frac{64\pi}{\sqrt{3}\cdot 2^\omega}\cdot\frac{P_M}{(lR_I)^\alpha}\\
&\le \frac{1}{l^{\alpha-1}}\cdot\frac{9\pi\cdot 2^{7-\omega}P_MR_I^2}{\sqrt{3}R_I^\alpha R^2}.
\end{aligned} \tag{5}$$

The second inequality is by Inequality (4) and the last inequality is by $R<\frac{R_I}{2}$. Then

$$\begin{aligned}
\Psi_u^{w\notin I_u} &= \sum_{l=1}^{\infty}\Psi_u^{R_l} \le \frac{9\pi\cdot 2^{7-\omega}P_MR_I^2}{\sqrt{3}R_I^\alpha R^2}\cdot\sum_{l=1}^{\infty}\frac{1}{l^{\alpha-1}}\\
&\le \frac{9\pi\cdot 2^{7-\omega}P_MR_I^2}{\sqrt{3}R_I^\alpha R^2}\cdot\frac{\alpha-1}{\alpha-2}\\
&\le \frac{(1-1/c)P_C}{\rho\beta R^\alpha}. \quad\Box
\end{aligned} \tag{6}$$

By the Markov inequality, with probability at least $1-\frac{1}{\rho}$, the interference at some node u caused by nodes outside I_u cannot exceed $\rho\Psi_u^{w\notin I_u}$. Then if v is the only sending node in $E_v^{R_I+R}$, i.e., v is the only sending node in I_u for every $u\in T_v$, by the above Claim, with probability at least $1-\frac{1}{\rho}$, the SINR at node u can be bounded as follows:

$$\frac{\frac{P_C}{d(u,v)^\alpha}}{N+\rho\Psi_u^{w\notin I_u}} \geq \frac{\frac{P_C}{R^\alpha}}{\frac{P_C}{c\beta R^\alpha}+\frac{(1-1/c)P_C}{\beta R^\alpha}} \geq \beta \tag{7}$$

Thus u can successfully receive the message sent from v according to the SINR constraint (1), which concludes the proof. □

Based on the sufficient condition for a successful transmission in Lemma 4, Lemma 5 in the following lists the successful transmissions of messages used in the algorithm in certain given timeslots with high probability. Then Lemma 6 states that, with high probability, a leader will not quit the algorithm until all its neighbors have been colored.

Lemma 5. *Assume that Property 1 and Property 2 hold. Then with probability at least $1-\frac{1}{n^4}$, the following results are correct:*

(i) After entering state $\mathcal{G}$, a node v can successfully send a StartColoring message to all its neighbors in $\mu_1 \log n$ timeslots.
(ii) A node v in state $\mathcal{G}$ can successfully send a Grant message to all its neighbors in $\mu_1 \log n$ timeslots.
(iii) A node v in state $\mathcal{G}$ can successfully send a StartTransmit message to all nodes within distance $3R$ in $\mu_2 \log n$ timeslots.
(iv) A node v in state $\mathcal{C}_2$, after choosing a color, can successfully send a Color$_v$ message to all neighbors in $\mu_1 \log n$ timeslots.

Proof. We prove only *(i)* here. *(ii)*, *(iii)*, *(iv)* can be proved in a manner very much similar to *(i)*.

Proof of (i): As shown in Lemma 4, if v is the only sending node in $E_v^{R_I+R}$, with probability $1-\frac{1}{\rho}$, the *StartColoring* message sent by v can be received successfully by all nodes in T_v. Let P_1 denote the event that v is the only sending node in $E_v^{R_I+R}$, then

$$\begin{aligned} P_1 &= 2^{-\omega}\prod_{u\in E_v^{R_I+R}\setminus\{v\}}(1-p_u) \\ &\geq 2^{-\omega}\prod_{u\in E_v^{R_I+R}}(1-p_u) \\ &\geq 2^{-\omega}\cdot\left(\frac{1}{4}\right)^{\sum_{u\in E_v^{R_I+R}} p_u} \\ &\geq 2^{-\omega}\cdot\left(\frac{1}{4}\right)^{3\cdot 2^{1-\omega}\cdot\chi(R_I+R,0.5R)} \end{aligned} \tag{8}$$

The last inequality is by Lemma 1 and Lemma 3. Then the probability P_{no} that v fails to transmit the *StartColoring* message to all nodes in T_v is at most

$$\begin{aligned} P_{no} &\leq \left(1-(1-1/\rho)2^{-\omega}\cdot\left(\frac{1}{4}\right)^{3\cdot 2^{1-\omega}\cdot\chi(R_I+R,0.5R)}\right)^{\mu_1\log n} \\ &\leq e^{-(1-1/\rho)2^{-\omega}\mu_1\log n\cdot(\frac{1}{4})^{3\cdot 2^{1-\omega}\cdot\chi(R_I+R,0.5R)}} \in n^{-4}. \end{aligned} \tag{9}$$

Lemma 6. *Assume that Property 1 and Property 2 hold. Then with probability at least $1-\frac{1}{n^4}$, a node v in state $\mathcal{G}$ will not quit the algorithm until all its neighbors have been colored.*

Proof. Assume that v quits the algorithm in timeslot t when there are still $d>0$ neighbors in state $\mathcal{C}_1$. Denote the set of these d nodes as T. We further assume that v forces d_v neighbors join state $\mathcal{C}_1$ after transmitting the *StartColoring$_v$* message. Thus before time t, v has transmitted (d_v-d) *Grant* messages. Then by Algorithm 1, v has not received a *RequestColor* message since the timeslot $t-((d_v-d)\cdot 5\mu_1\log n+3\mu_1\log^2 n+\mu_1\log n)$. Next we show that during the interval $[t-((d_v-d)\cdot 5\mu_1\log n+3\mu_1\log^2 n+\mu_1\log n),t)$, there is at least one node that can successfully transmit a *RequestColor* message to v with high probability. Then v will not quit the algorithm in timeslot t. This contradiction completes the proof.

By Algorithm 1, the initial transmission probability of each node in T is assigned as $\frac{2^{-\omega-1}}{n}$, and each node in T will either double its transmission probability every $3\mu_1\log n$ timeslots, or receive a *Grant* message from v and halve the transmission probability. Because v received the last *RequestColor* message before the timeslot $t-((d_v-d)\cdot 5\mu_1\log n+3\mu_1\log^2 n+\mu_1\log n)$ and v transmits each *Grant* message for $2\mu_1\log n$ timeslots, v has completed the transmission of (d_v-d) *Grant* messages by the timeslot $t-((d_v-d)\cdot 5\mu_1\log n+3\mu_1\log^2 n+\mu_1\log n)+2(d_v-d)\mu_1\log n-1$. So in timeslot $t^*=t-((d_v-d)\cdot 3\mu_1\log n+3\mu_1\log^2 n+\mu_1\log n)$, each node in T has transmission probability at least $\frac{2^{-\omega-1-d_v+d}}{n}$. From t^*, each node in T doubles its transmission probability every $3\mu_1\log n$ timeslots. In timeslot $t-\mu_1\log n$, each node in T has a constant transmission probability of $2^{-\omega-1}$. Then using a similar argument as in the proof of Lemma 5, we can get that with probability at least $1-n^{-4}$, there is at least one node in T that can successfully transmit a *RequestColor* message to v by the timeslot $t-1$. □

Lemma 7. *Assume Property 1 and Property 2 hold. A node v will correctly get a color after waking up for $O(\Delta \log n + \log^2 n)$ timeslots with probability $1-O(n^{-2})$.*

Proof. After waking up for at most $2(\mu_1+\mu_2)\log n$ timeslots, v enters state $\mathcal{S}$ or starts executing the MIS algorithm. If v takes part in the MIS algorithm, by Lemma 2, with probability $1-O(n^{-3})$, it will correctly enter state $\mathcal{S}$ or state $\mathcal{G}$ after $O(\log^2 n)$ timeslots. Next we bound the time v stays in states $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{G}$.

We first bound the time that node v would stay in state $\mathcal{C}_1$. Assume that u is the leader of v. By Algorithm 1, during every $3\mu_1 \log n$ timeslots, either v receives at least one new *Grant* message from u, or it doubles its transmission probability. If the received *Grant* message is not for v, it means that a node in $N(u)$ will join state $\mathcal{C}_2$. By Lemma 2, with probability $1-O(n^{-3})$, when u stays in state $\mathcal{M}$, there would not be another node in E_u^{3R} that is in state $\mathcal{M}$. By the MIS algorithm and Lemma 12 in Appendix A, with probability $1-O(n^{-4})$, u can force all other nodes in E_u^{3R} to join state $\mathcal{S}$ and not to continue competing for joining state $\mathcal{M}$ until receiving a *StartTransmit*$_u$ message from u. Thus, with probability $1-O(n^{-3})$, there are no other nodes in E_u^{3R} joining state $\mathcal{G}$ when u stays in state $\mathcal{G}$. Additionally, only nodes in $N(u)$ and $E_u^{3R} \setminus E_u^{2R}$ may join state $\mathcal{C}_1$ by receiving a *StartColoring* message before u quits. Thus all nodes in $E_u^{2R} \setminus N(u)$ will stay in state $\mathcal{S}$ while u stays in state $\mathcal{G}$. Then after at most $(2(\Delta-1)+\log n)3\mu_1 \log n$ timeslots, either v receives a *Grant*$_v$ message and joins state $\mathcal{C}_2$, or v has transmission probability of $2^{-1-\omega}$, since v can receive at most $\Delta-1$ *Grant* messages not for v and each of which would halve v's transmission probability. Then by a similar argument to that in Lemma 5, v will successfully transmit a *RequestColor* message to u in $2\mu_1 \log n$ timeslots with probability $1-n^{-4}$. Furthermore, by Lemma 6, with probability $1-n^{-4}$, u did not quit the algorithm before receiving the *RequestColor* message from v. After successfully transmitting message *RequestColor*$_v$ to u, by Algorithm 1 and Lemma 5*(ii)*, with probability $1-n^{-4}$, v will receive a *Grant*$_v$ message from u in at most $2\mu_1 \Delta \log n$ timeslots. So each node will stay in state $\mathcal{C}_1$ for at most $8\mu_1 \Delta \log n + 3\mu_1 \log^2 n$ timeslots with probability at least $1-O(n^{-3})$. By Algorithm 1, it is easy to see that each node stays in state $\mathcal{C}_2$ for $\mu_1 \log n$ timeslots.

Next we bound the time that a node v stays in state $\mathcal{G}$. By Lemma 5*(i)*, after entering state $\mathcal{G}$ for $\mu_1 \log n$ timeslots, v will successfully send a *StartColoring* message to all its neighbors with probability $1-n^{-4}$. Then all nodes in $N(v)$ without choosing their colors will enter state $\mathcal{C}_1$. As shown above, with probability at least $(1-O(n^{-3}))^{\Delta} \in 1-O(n^{-2})$, each node in $N(v)$ will join state $\mathcal{C}_2$ after joining state $\mathcal{C}_1$ for at most $O(\Delta \log n + \log^2 n)$ timeslots. Then by the algorithm, v will quit after waiting additionally for $O(\Delta \log n + \log^2 n)$ timeslots by noticing that b_v is at most Δ. So with probability at least $1-O(n^{-2})$, the total time that v stays in state $\mathcal{G}$ is at most $O(\Delta \log n + \log^2 n)$.

Next we bound the time from when v wakes up to its next entry to state $\mathcal{C}_1$ or $\mathcal{G}$. By the algorithm, after waking up for at most $2(\mu_1+\mu_2)\log n$ timeslots, either v starts executing the MIS algorithm or there comes a node in E_v^{3R} joining state $\mathcal{G}$. If v starts executing the MIS algorithm, by Lemma 2, with probability at least $1-O(n^{-3})$, there will be a node in E_v^{3R} joining state $\mathcal{G}$. So after waking up for at most $O(\log^2 n)$ timeslots, a node in E_v^{3R} will join state $\mathcal{G}$. From then on, by Algorithm 1 and the analysis above, with probability at least $1-O(n^{-2})$, after every $O(\Delta \log n + \log^2 n)$ timeslots, there will be at least one node u in E_v^{3R} joining state $\mathcal{G}$ and all nodes in $N[u]$ would quit the algorithm. We can see that all nodes joining state $\mathcal{G}$ are independent in terms of R. So there are only a constant number of nodes in E_v^{3R} being able to join state $\mathcal{G}$, denoted by $\hat{c}$. Then after at most $\hat{c}O(\Delta \log n + \log^2 n)$ timeslots, there will be a node in $N[v]$ joining state $\mathcal{G}$. Thus, with probability at least $1-O(n^{-2})$, the total time that v spends before entering state $\mathcal{C}_1$ or $\mathcal{G}$ after waking up is at most $O(\Delta \log n + \log^2 n)$.

Combining all the above, with probability $1-O(n^{-2})$, every node stays in the algorithm for at most $O(\Delta \log n + \log^2 n)$ timeslots. Finally, we prove that each node can correctly get a color with probability at least $1-O(n^{-2})$. As shown before, with probability $1-O(n^{-3})$, when a node v is in state $\mathcal{G}$, there cannot be another node in E_v^{3R} being in state $\mathcal{G}$. By Lemma 6, with probability $1-O(n^{-4})$, v will not leave state $\mathcal{G}$ until all its neighbors are colored. Thus, with probability $1-O(n^{-2})$, all nodes with color 0, i.e., all nodes used to join state $\mathcal{G}$, are independent in terms of R. If v chooses another color, by the algorithm, it will choose an available color and broadcast the chosen color to its neighbors as soon as it receives the *Grant* message from its leader. By Property 2*(i)*, there is not a node in $N(v)$ staying in state $\mathcal{C}_2$ when v is in state $\mathcal{C}_2$. By Lemma 5*(iv)*, when staying in state $\mathcal{C}_2$, v can successfully send its color to its neighbors with probability $1-n^{-4}$. Note also that in the coloring algorithm, by Lemma 12, with probability $1-n^{-4}$, v has been woken up before the first node in its neighborhood starts choosing a color. Thus when v chooses a color in state $\mathcal{C}_2$, with probability $1-n^{-3}$, v has received all the colors chosen by its neighbors and there are no other nodes in $N(v)$ choosing a color at the same time. So v will correctly select a color with probability $1-O(n^{-2})$. □

Lemma 8. *Property 2 holds with probability $1-O(n^{-1})$.*

Proof. We will show that with high probability, none of *(i)* *(ii)* and *(iii)* is the first property to be violated.

Claim. With probability at least $1-n^{-1}$, Property 2(i) is not the first property to be violated.

Proof. Assume that Property 2*(i)* is the first one to be violated, and D_i is the first disk violating it in timeslot t. We further assume that node $v \in D_i$ joins state $\mathcal{C}_2$ in timeslot t and another node u also stays in state $\mathcal{C}_2$ in timeslot t. Assume that w is u's leader. We can still assume that all properties are correct before t. By Algorithm 1, w starts transmitting $Grant_u$ after the timeslot $t-3\mu_1\log n$, since w transmits $Grant_u$ for $2\mu_1\log n$ timeslots and u stays in state $\mathcal{C}_2$ for $\mu_1\log n$ timeslots after receiving $Grant_u$. Assume that $Grant_u$ is the i-th *Grant* message transmitted by w. Then the message $RequestColor_u$ was received by w after the timeslot $t-(i-1)2\mu_1\log n-3\mu_1\log n$, since w transmits a *Grant* message every $2\mu_1\log n$ timeslots as long as Q_w is not empty. By Algorithm 1, w waits for $5i\mu_1\log n+3\mu_1\log^2 n+\mu_1\log n$ timeslots to decide whether it quits the algorithm after receiving $RequestColor_u$. So w will not quit by timeslot $t+\Omega(\log^2 n)$. Additionally, we can assume that all properties are correct before t. By Property 5 and Lemma 12 in Appendix A, after w joins state $\mathcal{M}$, it is the only node in E_w^{3R} staying in state $\mathcal{M}$ and it will force all nodes in E_w^{3R} to join state $\mathcal{S}$ with probability $1-O(n^{-4})$. By Algorithm 1, all these nodes will not try to compete for joining state $\mathcal{G}$ until they receive a $StartTransmit_w$ from w. Then when w stays in state $\mathcal{G}$, with probability $1-O(n^{-4})$, there cannot be another node in E_w^{3R} staying in state $\mathcal{G}$. Consequently, w must also be v's leader with probability $1-O(n^{-4})$. Furthermore, w must have started transmitting $Grant_v$ before timeslot t. Hence, by Algorithm 1, w must have started transmitting $Grant_u$ by timeslot $t-2\mu_1\log n$. Then by Lemma 5*(ii)*, u has received $Grant_u$ from w by $t-\mu_1\log n-1$ with probability at least $1-n^{-4}$. Because u stays in state C_2 for $\mu_1\log n$ timeslots, u must have quit the algorithm before t with probability at least $1-n^{-4}$. By this contradiction, Property 2*(i)* is not the first violated property when u stays in state C_2 with probability $1-n^{-3}$. Then for D_i, Property 2*(i)* is not the first violated one when there is a node in D_i staying in state C_2 with probability $1-n^{-2}$. And the lemma is correct for every disk with probability at least $1-n^{-1}$. □

Claim. With probability at least $1-n^{-1}$, Property 2*(ii)* is not the first property to be violated.

Proof. Assume that Property 2*(ii)* is the first property to be violated, and that D_i is the first disk violating the property in timeslot t^*. Before timeslot t^*, we can still assume that all properties hold. Assume that v is the leader of some nodes of D_i that stays in $\mathcal{C}_1$. Denote $C_{v1}(t)$ as the set of nodes in $N(v)$ that are in state $\mathcal{C}_1$ in timeslot t. By Property 5 in Appendix A, we know that in any timeslot before t^*, all nodes in state $\mathcal{M}$ constitute an independent set in terms of $3R$. By Algorithm 1 and Algorithm MIS, each node in state $\mathcal{M}$ will join state $\mathcal{G}$ after transmitting a *DoNotTransmit* message. By Lemma 12 in Appendix A, the $DoNotTransmit_v$ message sent by v can be received by all nodes in E_v^{3R} with probability at least $1-O(n^{-4})$. All these nodes will not restart competing for joining state $\mathcal{G}$ until they receive a $StartTransmit_v$ message from v. Thus, with probability $1-O(n^{-4})$, there is not another node in E_v^{3R} staying state $\mathcal{G}$ when v stays in $\mathcal{G}$. Additionally, because all properties still hold before t^* and there are some neighbors of v staying in state $\mathcal{C}_1$ by t^*, by Lemma 6, v will not leave the state $\mathcal{G}$ by timeslot t^*-1 with probability $1-O(n^{-4})$. So v does not start transmitting the $StartTransmit_v$ message before timeslot $t^*-\mu_2\log n-1$. Also, noticing that each node in E_v^{3R} needs $\Omega(\log^2 n)$ timeslots to join state $\mathcal{G}$ by executing Algorithm MIS, there will be no other nodes in E_v^{3R} joining state $\mathcal{G}$ by timeslot $t^*+\Omega(\log^2 n)$. Thus in timeslot t^*, with probability at least $1-O(n^{-4})$, all nodes in D_i that are in state $\mathcal{C}_1$ have the same leader v. Next we prove a slightly stronger result: with probability at least $1-O(n^{-2})$, in any timeslot t, the sum of transmission probabilities of all nodes in $C_{v1}(t)$ is at most $2^{-\omega}$. Then during v's stay in state $\mathcal{G}$, there exists no such a timeslot t^* for D_i with probability at least $1-O(n^{-2})$.

Otherwise, assume that in timeslot t, $\sum_{u\in C_{v1}(t)} p_u > 2^{-\omega}$. Denote $I=[t-3\mu_1\log n, t)$. By Algorithm 1, every node in C_{v1} doubles its transmission probability at most once during the interval. Furthermore, some nodes in state $\mathcal{S}$ may join state $\mathcal{C}_1$ during the interval. However, the sum of transmission probabilities of newly joined nodes is at most $\frac{2^{-\omega-1}}{n}\cdot n = 2^{-\omega-1}$. Hence, it holds that in timeslot $t-3\mu_1\log n$, the sum of transmission probabilities is at least $2^{-2-\omega}$. Consequently, before any violation timeslot, there is an interval I such that $2^{-2-\omega} \le \sum_{u\in C_{v1}} p_u < 2^{-\omega}$. Because Property 2*(ii)* is the first violated one, we can still assume that other properties are correct. So during the interval I, for any disk D_j, $j\neq i$, $\sum_{v\in D_j} p_v \le 3\cdot 2^{1-\omega}$.

Next we show that with probability at least $1-n^{-4}$, v will successfully send a new *Grant* message to all its neighbors during the interval $(t-3\mu_1\log n, t)$. Clearly, if all nodes in $C_{v1}(t-3\mu_1\log n)$ join state $\mathcal{C}_2$ by $t-1$, then $\sum_{u\in C_{v1}(t)} p_u$ is at most the sum of transmission probabilities of newly joined nodes. As discussed above, it is at most $2^{-\omega-1}$. So in the following, it can be assumed that not all nodes in $C_{v1}(t-3\mu_1\log n)$ have joined state $\mathcal{C}_2$ by time $t-1$.

We claim that at least one node in C_{v1} can send a message *RequestColor* to v during the interval $I_1=[t-3\mu_1\log n, t-2\mu_1\log n-1]$. Using a similar argument as in Lemma 4, if a node $w\in N(v)$ is the only transmitting node in $E_v^{R_I}$, then v can receive the message from w successfully with probability at least $1-1/\rho$. Denote D as a minimum cover of disks with radius $\frac{R}{2}$ for $E_v^{R_I}$. Then in any timeslot during I_1, the probability P_{only} that there is only one node $w\in C_{v1}$ transmitting is

$$\begin{aligned}
P_{only} &= \sum_{w\in C_{v1}} p_w \prod_{w'\in E_u^{R_I}\setminus\{v\}} (1-p_{w'}) \\
&\geq \sum_{w\in C_{v1}} p_w \prod_{D_j\in D} \prod_{w'\in D_j} (1-p_{w'}) \\
&\geq \sum_{w\in C_{v1}} p_w \prod_{D_j\in D} \left(\frac{1}{4}\right)^{\sum_{w'\in D_j} p_{w'}} \\
&\geq \sum_{w\in C_{v1}} p_w \left(\frac{1}{4}\right)^{\chi(R_I,R/2)\sum_{w'\in D_j} p_{w'}} \\
&\geq 2^{-\omega-2}\cdot\left(\frac{1}{4}\right)^{\chi(R_I,R/2)\cdot 3\cdot 2^{1-\omega}}.
\end{aligned} \tag{10}$$

The last inequality is by Lemma 3. So during I_1, the probability P_T that there is not any node successfully transmitting a *RequestColor* message to v is at most

$$P_T \leq \left(1-\frac{1}{2}\left(1-\frac{1}{\rho}\right)\cdot 2^{-\omega-2}\left(\frac{1}{4}\right)^{\chi(R_I,R/2)\cdot 3\cdot 2^{1-\omega}}\right)^{\mu_1\log n} \in O\left(n^{-4}\right). \tag{11}$$

Thus with probability at least $1-O(n^{-4})$, v receives a *RequestColor* message during the interval I_1. Denote t_1 as the first timeslot when v starts broadcasting a new *Grant* message after $t-3\mu_1\log n$. Because v broadcasts a *Grant* message for $2\mu_1\log n$ timeslots and v receives a *RequestColor* message by $t-2\mu_1\log n$, such a timeslot exists in the interval $(t-3\mu_1\log n, t-\mu_1\log n]$ with probability at least $1-n^{-4}$. Then by Lemma 5*(ii)*, during the interval $(t-3\mu_1\log n, t-1]$, with probability at least $1-n^{-4}$, all nodes in C_{v1} receive a new $Grant_w$ message and halve their transmission probabilities, except for w, which enters state $\mathcal{C}_2$. Denote t_2 as the first timeslot when all nodes in C_{v1} have received the new *Grant* message. By Algorithm 1, all these nodes will not increase the transmission probability until $t_2+3\mu_1\log n-(t_2-t_1)=t_1+3\mu_1\log n>t$. Note that before halving the transmission probability, $\sum_{u\in C_{v1}} p_u \leq 2^{-\omega}$. So after halving the transmission probability, the sum is at most $2^{-1-\omega}$ for these nodes. Also notice that all newly joined nodes have transmission probability sum at most $2^{-1-\omega}$. So during the interval $[t_1, t_1+3\mu_1\log n)$, $\sum_{u\in C_{v1}} p_u \leq 2^{-1-\omega}+2^{-1-\omega}=2^{-\omega}$. Thus with probability at least $1-O(n^{-4})$, D_i will not violate Property 2*(ii)* in timeslot t.

From the above, we know that in the first $O(n^2)$ timeslots when v stays in state $\mathcal{G}$, with probability $1-O(n^{-2})$, there is not a timeslot such that Property 2*(ii)* is the first one to be violated in D_i. By Lemma 7, v stays in state $\mathcal{G}$ for at most $O(\Delta\log n+\log^2 n)$ timeslots with probability at least $1-O(n^{-2})$. Thus when v stays in state $\mathcal{G}$, there is not a violation timeslot for D_i with probability at least $1-O(n^{-2})$. Additionally, when there are nodes in D_i which are in state $\mathcal{C}_1$, it means that there is a node staying in $E_i^{\frac{3R}{2}}$ in state $\mathcal{G}$. From Algorithm 1, we know that all nodes that joined state $\mathcal{G}$ during executing the algorithm are independent in terms of R. Hence, there are at most a constant number of nodes in $E_i^{\frac{3R}{2}}$ which can join state $\mathcal{G}$. Thus D_i is not the first disk violating Property 2*(ii)* with probability $1-O(n^{-2})$. Then Property 2*(ii)* is not the first violated property for all disks with probability at least $1-O(n^{-1})$. □

Claim. With probability at least $1-O(n^{-2})$, Property 2*(iii)* is not the first property to be violated.

Proof. Assume that *(iii)* is the first property to be violated, and D_i violates it in timeslot t for the first time. Then there is a new node u in D_i joining state $\mathcal{G}$ in timeslot t, while there has been another node v in D_i staying in state $\mathcal{G}$ in timeslot t. Before t, we can still assume that all properties are correct, since Property 2*(iii)* is the first one to be violated. By Property 5 in Appendix A, all nodes in state $\mathcal{M}$ constitute an independent set in any timeslot before t^*. Also, by Lemma 12 in Appendix A, after v joins state $\mathcal{M}$, it can successfully transmit a *DoNotTransmit*$_v$ message to all nodes in E_v^{3R} with probability $1-O(n^{-4})$. After that, by Algorithm 1, each node in E_v^{3R} will not try to join state $\mathcal{G}$ until it receives *StartTransmit*$_v$ from v. By Algorithm 1, v has not started transmitting *StartTransmit*$_v$ by the timeslot $t-\mu_2\log n$, since v still stays in state $\mathcal{G}$ in timeslot t. Also notice that each node needs $\Omega(\log^2 n)$ timeslots to join state $\mathcal{G}$ by executing Algorithm MIS. So there will not exist another node in E_v^{3R} joining state $\mathcal{G}$ by timeslot $t+\Omega(\log^2 n)$ with probability $1-O(n^{-4})$. This contradicts the fact that u joins state $\mathcal{G}$ in timeslot t. Thus when v stays in state $\mathcal{G}$, there is not such a violation timeslot t with probability $1-O(n^{-4})$. Clearly, each node can join state $\mathcal{G}$ for at most once. Then with probability $1-O(n^{-3})$, there is not a timeslot such that Property 2*(iii)* is first violated in D_i. This is true for every disk with probability $1-O(n^{-2})$. □

Theorem 1. *After waking up for $O(\Delta\log n+\log^2 n)$ timeslots, every node v will correctly get a color from $\{0,1,\ldots,\Delta_v\}$ with probability at least $1-O(n^{-1})$.*

Proof. Since Properties 1 and 2 have been shown to be correct with probability $1 - O(n^{-1})$, by Lemma 7, with probability at least $1 - O(n^{-1})$, every node v will correctly choose a color after executing Algorithm 1 for at most $O(\Delta \log n + \log^2 n)$ timeslots. Furthermore, when v chooses a color, either v chooses color 0, or it chooses the first available color in its color list by Algorithm 1. Because v receives at most $\Delta_v - 1$ colors from its neighbors (one of its neighbors is a leader), v can still choose a color from $\{0, 1, \ldots, \Delta_v\}$. □

4. Distributed $(\Delta + 1)$-coloring for uniform power assignment

In some multi-hop radio networks, nodes may not be able to adjust their transmission powers. For these networks, assuming that nodes adopt uniform power assignment, i.e., all nodes transmit with the same power level, we can obtain a distributed $(\Delta + 1)$-coloring algorithm by iteratively carrying out the MIS algorithm. We only need to change the operations of the MIS algorithm in the last state $\mathcal{M}$. Each node in state $\mathcal{M}$ first chooses an available color that has not been chosen by its neighbors, and then transmits a message m_C containing its choice to its neighbors for $\mu_1 \log n$ timeslots with constant probability after waking up all its neighbors. Then all the nodes having received the message m_C delete the received color from their color list and resume executing the algorithm. By Lemma 2, we know that with high probability, in any timeslot, all nodes in state $\mathcal{M}$ form an independent set. Furthermore, similar to the proof of Lemma 5, we can show that with high probability, each node can successfully transmit its choice to its neighbors before any neighbor starts choosing a color. These two facts ensure the correctness of the computed coloring. We assume that all nodes transmit with power $P = cN\beta R^{\alpha}$. Then we can get the following lemma, based on which the theorem on the correctness and the time complexity of the proposed coloring algorithm can be proved.

Lemma 9. *With probability at least $1 - O(n^{-2})$, a node v will correctly get a color in $O(\Delta_v^{2R} \log^2 n)$ timeslots after starting executing the algorithm, where Δ_v^{2R} is the number of nodes in E_v^{2R}. Furthermore, v will choose a color from $\{0, 1, \cdots, \Delta_v\}$.*

Proof. Using a similar argument to that in the analysis of the MIS algorithm (in Appendix A), we can get that after a node v starts or restarts the algorithm for $O(\log^2 n)$ timeslots, there will be a node in E_v^{2R} joining state $\mathcal{M}$ with probability $1 - O(n^{-3})$. Thus after at most $O(\Delta_v^{2R} \log^2 n)$ timeslots, v will join state $\mathcal{M}$ with probability at least $1 - O(n^{-2})$. Furthermore, in a manner similar to proving Lemma 5, we can show that all neighbors of v which have chosen colors before v have informed v of their choices with probability $1 - O(n^{-3})$. And by Lemma 2, when v is in state $\mathcal{M}$, with probability $1 - O(n^{-3})$, none of v's neighbors stay in state $\mathcal{M}$ simultaneously. Thus v will correctly choose a color different from all its neighbors with probability at least $1 - O(n^{-3})$. All together, we know that with probability at least $1 - O(n^{-2})$, v will correctly get a color in $O(\Delta_v^{2R} \log^2 n)$ timeslots after starting executing the algorithm. Finally, since there are Δ_v nodes in v's neighborhood, v has deleted at most Δ_v different colors from its color list when v chooses a color. Thus v can choose a color from $\{0, 1, \cdots, \Delta_v\}$. □

Theorem 2. *If nodes adopt the uniform power assignment, there exists a distributed algorithm such that with probability at least $1 - O(n^{-1})$, each node will correctly get a color after executing the algorithm for $O(\Delta \log^2 n)$ timeslots. Furthermore, the total number of colors used is at most $\Delta + 1$.*

Proof. By Lemma 9, a node v, with probability at least $1 - O(n^{-2})$, will correctly get a color in $O(\Delta_v^{2R} \log^2 n)$ timeslots after starting executing the algorithm, where Δ_v^{2R} is the number of nodes in E_v^{2R}. Furthermore, v will choose a color from $\{0, 1, \cdots, \Delta_v\}$. Thus the theorem is correct for all nodes with probability $1 - O(n^{-1})$ by noting that $\Delta_v^{2R} \le \chi(2R, R)\Delta \in O(\Delta)$.

5. Conclusion

In this paper, we study the distributed $(\Delta + 1)$-coloring problem in unstructured multi-hop radio networks under the SINR interference model. Without relying on any knowledge about the neighborhood, our proposed new distributed $(\Delta + 1)$-coloring algorithm has time complexity $O(\Delta \log n + \log^2 n)$. Our result even matches the $O(\Delta)$-coloring algorithm in [4] for large Δ; their algorithm needs a prior estimate of Δ. For networks in which the nodes cannot adjust their transmission powers, we give a $(\Delta + 1)$-coloring algorithm with time complexity $O(\Delta \log^2 n)$. Furthermore, by carefully tuning the parameters, we show that the maximal independent set algorithm in [16] still works under the SINR constraint, which is of independent interest.

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Appendix A. Analysis of the MIS algorithm

In this appendix, we show that in an asynchronous circumstance, as long as the sum of transmission probabilities of nodes in any local region that are executing other algorithms can be upper bounded by a constant, the MIS algorithm given in Algorithm 2 is still correct and have the same asymptotically time complexity as that in [16] under the SINR model. Here we assume that in any disc with radius $\frac{R_{mis}}{2}$, the sum of transmission probabilities of nodes that are executing other algorithms is at most ϕ, where ϕ is a constant. By Property 2 and Lemma 1, in the coloring algorithm, ϕ is at most $3 \cdot 2^{-\omega} \cdot \chi(\frac{3R}{2}, \frac{R}{2})$ based on the setting $R_{mis} = 3R$.

By the assigned transmission power $P_{mis} = cN\beta R_{mis}^{\alpha}$ and the definition of the transmission range in Section 2, we can get that each node's transmission range is R_{mis}. Before the analysis, we define a parameter R_M as follows whose functionality is the same as R_I defined in Eq. (3).

$$R_M = R_{mis}\left(2^{6-\omega}\sqrt{3}\pi\eta\beta\left(3 \cdot 2^{-\omega} + \phi\right) \cdot \frac{1}{1 - 1/c} \cdot \frac{\alpha - 1}{\alpha - 2}\right)^{1/(\alpha-2)}, \tag{12}$$

where η is chosen such that $R_M > 2R_{mis}$. Furthermore, in this section, the notations T_i, D_i and I_i denote the disks centered at node i with radius R_{mis}, $\frac{R_{mis}}{2}$ and R_M, respectively. And E_i^r still denotes the disk centered at node i with radius r. By $\mathcal{A}_i$ we denote the set of nodes in D_i which are in state $\mathcal{A}$. $\mathcal{B}_i$ and $\mathcal{W}_i$ are defined similarly.

In order to ensure that the MIS algorithm in Algorithm 2 is correct with high probability, We define the constant parameters as follows:

$$\omega = 6.4, \qquad \mu_2 = \frac{2^{\omega+2} \cdot 4^{(3\cdot 2^{-\omega}+\phi)\cdot\chi(R_M+R_{mis},0.5R_{mis})}}{1 - 1/\eta}, \qquad \delta = \frac{720e\mu_2 \cdot 4^{(3\cdot 2^{-\omega}+\phi)\cdot\chi(R_M+2R_{mis},0.5R_{mis})}}{1 - 1/\eta},$$

$$\lambda = \frac{5 \cdot 2^{\omega+2} \cdot 4^{(3\cdot 2^{-\omega}+\phi)(\chi(R_M+R_{mis},0.5R_{mis})+1)}}{1 - \frac{1}{\eta}}, \qquad \kappa = \frac{180e \cdot 4^{(3\cdot 2^{-\omega}+\phi)\cdot\chi(R_M+R_{mis},0.5R_{mis})}}{1 - 1/\eta}, \qquad \tau = \kappa^{-1}.$$

Furthermore, θ is the size of a maximum independent set in terms of R in a disk with radius $2R$. By Lemma 1, we can get that $\theta \le 25$.

In the sequel, we will show the following three properties are correct with high probability. Property 3 states that the sum of transmitting probabilities by nodes in state $\mathcal{A}$ is bounded by a constant which helps to bound the probabilistic interference at some node. Property 4 states that the number of nodes in state $\mathcal{B}$ in a certain area of the network is bounded. Property 5 states that in any timeslot, the nodes in state $\mathcal{M}$ form an independent set.

Property 3. *For any disk D_i and in any timeslot t throughout the execution of the algorithm, $\sum_{v\in A_i} q_v(t) \le 2^{-\omega}$.*

Property 4. *For any disks D_i and in any timeslot t throughout the execution of the algorithm, $|\mathcal{B}_i| \le \kappa \log n$.*

Property 5. *In any timeslot during the execution of the algorithm, all nodes in state $\mathcal{M}$ constitute an independent set.*

Based on the above three properties, Property 1 can be easily obtained, since $R_{mis} > R$ and nodes in state $\mathcal{W}$ do not transmit. Furthermore, we can bound the sum of transmission probabilities of nodes in any disk D_i. In the following, we use p_v to denote the transmitting probability of node v.

Lemma 10. *Assume that Properties 3, 4 and 5 hold. For any disk D_i and in any timeslot t throughout the execution of the algorithm, the sum of transmission probabilities can be bounded as $\sum_{v\in D_i} p_v \le 3 \cdot 2^{-\omega} + \phi$.*

Using a similar method as in proving Lemma 4, we can get a similar sufficient condition for successful transmissions.

Lemma 11. *Assume Properties 3, 4 and 5 hold. If node v is the only sending node in $E_v^{R_M+R_{mis}}$, with probability $1 - \frac{1}{\eta}$, the message sent by v will be received successfully by all nodes in T_v.*

Similar to the proof of Lemma 5*(i)*, the following lemma can be obtained which states that a node in state $\mathcal{M}$ can successfully transmit a message to all nodes in T_v in $\mu_2 \log n$ timeslots.

Lemma 12. *Assume that Properties 3, 4 and 5 hold. Then after node v joins state $\mathcal{M}$, it will successfully transmit a waking-up message and the DoNotTransmit$_v$ message to all nodes in T_v in $2\mu_2 \log n$ timeslots with probability $1 - n^{-4}$.*

Proof. As shown in Lemma 11, if v is the only sending node in $E_v^{R_M+R_{mis}}$, with probability $1 - \frac{1}{\eta}$, the message sent by v will be received successfully by all nodes in T_v. Let P_1 denote the event that v is the only sending node in $E_v^{R_M+R_{mis}}$, then

$$P_1 \geq q_C \prod_{u \in E_v^{R_M+R_{mis}} \setminus \{v\}} (1-p_u) \geq 2^{-\omega} \prod_{u \in E_v^{R_M+R_{mis}}} (1-p_u)$$
$$\geq 2^{-\omega} \cdot \left(\frac{1}{4}\right)^{\sum_{u \in E_v^{R_M+R_{mis}}} p_u}$$
$$\geq 2^{-\omega} \cdot \left(\frac{1}{4}\right)^{(3 \cdot 2^{-\omega}+\phi)\cdot \chi(R_M+R_{mis}, 0.5R_{mis})} \tag{13}$$

The last inequality is by Lemma 1 and Lemma 10. Then the probability P_{no} that v fails to transmit a message to all nodes in T_v in $\mu_2 \log n$ timeslots is at most

$$P_{no} \leq \left(1-(1-1/\eta)2^{-\omega} \cdot \left(\frac{1}{4}\right)^{(3\cdot 2^{-\omega}+\phi)\cdot\chi(R_M+R_{mis},0.5R_{mis})}\right)^{\mu_2 \log n}$$
$$\leq e^{-(1-1/\eta)2^{-\omega}\mu_2 \log n \cdot (\frac{1}{4})^{(3\cdot 2^{-\omega}+\phi)\cdot\chi(R_M+R_{mis},0.5R_{mis})}} \in n^{-4}. \quad \square \tag{14}$$

Based on the sufficient condition for successful transmissions in Lemma 11, the remaining analysis of the MIS algorithm is very similar to that in [16].

Lemma 13. *Assume that Properties 3, 4 and 5 hold. Let t_v be the timeslot when v joins state $\mathcal{B}$. Then during the interval $[t_v - 2\mu_2 \log n, t_v + 2\delta \log^2 n]$, with probability at least $1 - O(n^{-4})$, there will be a node in $E_v^{2R_{mis}}$ joining state $\mathcal{M}$.*

Proof. If v receives a message $DoNotTransmit_u$ during the interval, by Lemma 12, u entered state $\mathcal{M}$ after the time $t_v - 2\mu_2 \log n$, since otherwise, with probability at least $1 - O(n^{-4})$, v would have received the message before t_v and would not join state $\mathcal{B}$, which contradicts the definition of t_v. In the following, we suppose that v does not receive messages *DoNotTransmit* during the interval.

First, we define some notations that will be used in the proof. A node in state $\mathcal{B}$ is called active if its *step* is larger than $2\mu_2 \log n$. For node v, we call a timeslot active if there is at least one active node in T_v. Otherwise, the timeslot is called inactive. Then we bound the number of active timeslots before an active node in T_v can send successfully, i.e., the message sent by this node can be successfully received by all nodes in its transmission range. Denote P as the event that an active node $w \in T_v$ sends successfully in an active timeslot. By Lemma 11, if a node $w \in T_v$ is the only sending node in $E_v^{R_M+2R_{mis}}$, with probability at least $1 - \frac{1}{\eta}$, the message $m_B(count_w)$ sent by w will be received successfully by all nodes in T_w. Then using a similar argument in Inequality (13), by Lemma 10, we can obtain that $P \geq (1-1/\eta)\frac{\tau}{2^\omega \log n} \cdot (\frac{1}{4})^{(3\cdot 2^{-\omega}+\phi)\cdot\chi(R_M+2R_{mis},R_{mis}/2)}$.

Then the probability P_f that there is no active node sending successfully in $\frac{\delta}{2}\log^2 n$ active timeslots is at most

$$P_f \leq \left(1-(1-1/\eta)\frac{\tau}{2^\omega \log n} \cdot \left(\frac{1}{4}\right)^{(3\cdot 2^{-\omega}+\phi)\cdot\chi(R_M+2R_{mis},R_{mis}/2)}\right)$$
$$\leq e^{-(1-1/\eta)\frac{\delta}{2}\log^2 n \cdot \frac{\tau}{2^\omega \log n} \cdot (\frac{1}{4})^{(3\cdot 2^{-\omega}+\phi)\cdot\chi(R_M+2R_{mis},R_{mis}/2)}} \in O\left(n^{-4}\right). \tag{15}$$

Thus with probability at least $1 - O(n^{-4})$, the number of active timeslots before an active node in T_v can send successfully is at most $\frac{\delta}{2}\log^2 n$.

Next we bound the number of inactive timeslots before an active node in T_v can send successfully. In a timeslot t, by Lemma 11, the probability P_1 that there is an active node $w \in T_v$ transmitting successfully is lower bounded as follows

$$P_1 \geq (1-1/\eta)n(t)q_B \prod_{u \in E_v^{R_M+2R_{mis}} \setminus \{w\}} (1-p_u)$$
$$\geq (1-1/\eta)n(t)q_B \prod_{u \in E_v^{R_M+2R_{mis}}} (1-p_u)$$
$$\geq (1-1/\eta)n(t)q_B \cdot \left(\frac{1}{4}\right)^{\sum_{u \in E_v^{R_M+2R_{mis}}} p_u}$$
$$\geq (1-1/\eta)n(t)q_B \cdot \left(\frac{1}{4}\right)^{(3\cdot 2^{-\omega}+\phi)\cdot\chi(R_M+2R_{mis},R_{mis}/2)}, \tag{16}$$

where $n(t)$ is the number of active nodes in T_v. Then the conditional probability $P[suc|send]$ which means an active node in T_v sends successfully when at least one active nodes in T_v sends is at least

$$P[suc|send] = \frac{P[suc \cap send]}{P[send]} = \frac{P_1}{P[send]} \geq (1 - 1/\eta) \cdot \left(\frac{1}{4}\right)^{(3 \cdot 2^{-\omega} + \phi) \cdot \chi(R_M + 2R_{mis}, R_{mis}/2)}, \tag{17}$$

Thus if at least one active node in T_v transmits, there is a node sending successfully with constant probability. Then if v resets its *count*, which means that there is at least one active node in T_v sending message $m_B(count)$, with constant probability, an active node in T_v sends successfully. Then we upper bound the probability P_m that the number of timeslots with non-successful transmissions which reset v's *count* is more than $\frac{\delta}{4\mu_2} \log n$ as follows

$$\begin{aligned} P_m &\leq \left(1 - (1 - 1/\eta) \cdot \left(\frac{1}{4}\right)^{(3 \cdot 2^{-\omega} + \phi) \cdot \chi(R_M + 2R_{mis}, R_{mis}/2)}\right)^{\frac{\delta}{4\mu_2} \log n} \\ &\leq e^{\frac{\delta}{4\mu_2} \log n \cdot (1 - 1/\eta) \cdot (\frac{1}{4})^{(3 \cdot 2^{-\omega} + \phi) \cdot \chi(R_M + 2R_{mis}, R_{mis}/2)}} \in O\left(n^{-4}\right). \end{aligned} \tag{18}$$

Furthermore, by Algorithm 2, after v resets its counter, there are at most $2\mu_2 \log n$ inactive timeslots for T_v. Thus with probability $1 - O(n^{-4})$, the number of inactive timeslots before an active node in T_v sends successfully is at most $\frac{\delta}{4\mu_2} \log n \cdot 2\mu_2 \log n = \frac{\delta}{2} \log^2 n$.

By the time $t_v + \delta \log^2 n$, either the number of active timeslots or the number of inactive timeslots is at least $\frac{\delta}{2} \log^2 n$. From above, we know that in either case, there will be an active node in T_v sending successfully with probability $1 - O(n^{-4})$. Assume such a node is w. By time $t_v + \delta \log^2 n$, w has informed all nodes in T_w about its *count* and makes these nodes reset their *count*. After that there are no neighbors being able to reduce w' *count*. The only way to prevent w from entering state $\mathcal{M}$ by the time $t_v + 2\delta \log^2 n$ is that some node in state $\mathcal{M}$ sends a message to w. Thus by the time $t_v + 2\delta \log^2 n$, w or one of its neighbors will join state $\mathcal{M}$ with probability at least $1 - O(n^{-4})$. □

Lemma 14. *Assume that Properties 3, 4 and 5 hold. For a node v, let t_v be a timeslot that v joins state $\mathcal{B}$. Then after $4\theta\delta \log^2 n + 2\mu_2 \log n$ timeslots, v will join state $\mathcal{M}$ or $\mathcal{S}$ with probability at least $1 - O(n^{-4})$.*

Proof. After t_v, by repeatedly using Lemma 13, with probability $1 - O(n^{-4})$, there will be a node in $E_v^{2R_{mis}}$ entering state $\mathcal{M}$ in every internal $[t_v + 2r\delta \log^2 n - 2\mu_2 \log n, t_v + 2(r+1)\delta \log^2 n]$ for $r \geq 0$. Thus by the time $t_v + 4\theta\delta \log^2 n$, there will be at least θ nodes in $E_v^{2R_{mis}}$ joining state $\mathcal{M}$ with probability at least $(1 - O(n^{-4}))^\theta \in 1 - O(n^{-4})$, since every such node can cover at most two adjacent intervals. Furthermore, by Lemma 12, every such node w will force all nodes in T_w to join state $\mathcal{S}$ with probability at least $1 - O(n^{-4})$. Thus all nodes joining state $\mathcal{M}$ constitute an independent set in $E_v^{2R_{mis}}$. Because the size of an independent set in $E_v^{2R_{mis}}$ is at most θ, one of these θ nodes is in T_v. If it is not node v, by Lemma 12, with probability $1 - O(n^{-4})$, v will receive the message *DoNotTransmit* from this node in $2\mu_2 \log n$ timeslots after this node joins state $\mathcal{M}$. Hence, by the time $t_v + 4\theta\delta \log^2 n + 2\mu_2 \log n$, v will have joined state $\mathcal{M}$ or state $\mathcal{S}$ with probability at least $1 - O(n^{-4})$. □

Based on Lemma 11 and using a similar idea as that in [16], the following lemma can be obtained. Before that, a clearance is defined as follows.

Definition 2. Let t be a timeslot in which a message m_A is sent by a node $v \in \mathcal{A}_i$ and received by all nodes $w \in D_i \setminus \{v\}$ without collision. Such a timeslot is called a clearance of D_i.

Lemma 15. *Properties 3, 4, 5 hold with probability at least $1 - O(n^{-3})$.*

Proof. The idea of proving Lemma 15 is similar to that in [16]. However, our analysis is under the SINR model, which is therefore more challenging. By showing that none of Properties 3, 4, 5 is the first one to be violated, we complete the proof of Lemma 15.

Claim. Assume Property 3 is the first property to be violated and let t_1 be the first time-slot in which the violation occurs. The probability that there exists such a time-slot t_1 during the execution of the algorithm is at most $P_{fail} \in O(n^{-3})$.

Proof. By assumption, at time t_1, the first violation occurs in a disk D_i, which means that $\sum_{v \in \mathcal{A}_i} q_v(t_1 - 1) \leq 2^{-\omega}$ and $\sum_{v \in \mathcal{A}_i} q_v(t_1) > 2^{-\omega}$. Consider the interval $\mathcal{I} = [t_1 - \lambda \log n, \ldots, t_1]$. By the definition of the algorithm, every node $v \in \mathcal{A}_i$ doubles its sending probability q_v exactly once during the interval $\mathcal{I}$. Furthermore, some node that were previously in $\mathcal{W}_i$ may join the set $\mathcal{A}_i$, but the total sending probabilities of these nodes is at most $n \cdot \frac{2^{-\omega-1}}{n} = 2^{-\omega-1}$ according to the initial probability setting. Then the sum of sending probabilities at time $t_1 - \lambda \log n$ is at least

$$\sum_{v\in\mathcal{A}_i} q_v(t_1-\lambda\log n) \geq \frac{1}{2}\left(2^{-\omega}-2^{-\omega-1}\right) = 2^{-\omega-2}. \tag{19}$$

Consequently, if Property 3 is violated, there must be an interval $\mathcal{I}$ preceding the violation during which the sum of sending probabilities is in the range

$$2^{-\omega-2} \leq \sum_{v\in\mathcal{A}_i} q_v(t) \leq 2^{-\omega} \quad \forall t\in\mathcal{I}. \tag{20}$$

Since t_1 is the first time-slot violating Property 3, in all disks $D_j \in E_i^{R_M+0.5R_{mis}}$, the sum of sending probabilities is

$$0 \leq \sum_{v\in\mathcal{A}_j} q_v(t) \leq 2^{-\omega} \quad \forall t\in\mathcal{I}. \tag{21}$$

Next we show that with high probability, a clearance occurs in the interval $\mathcal{I}$. Let P_{no} denote the probability that in a given time-slot $t\in\mathcal{I}$ no node in $E_i^{R_M+0.5R_{mis}} \setminus D_i$ sends. By P_{one} we denote the probability that exactly one node $v\in\mathcal{A}_i$ sends in t. Clearly, for each node u in D_i, $I_u \in E_i^{R_M+0.5R_{mis}}$. Similar to that in proving Lemma 4, the probabilistic interference caused by nodes outside $E_i^{R_M+0.5R_{mis}}$ is at most $\frac{(1-1/c)P}{\beta R^\alpha}$ with probability at least $1-1/\eta$. Then if only one node in D_i sends and no node in $E_i^{R_M+0.5R_{mis}} \setminus D_i$ sends, the SINR at node u is at least

$$\frac{\frac{P_{mis}}{R_{mis}^\alpha}}{N+\frac{(1-1/c)P_{mis}}{\beta R_{mis}^\alpha}} = \frac{\frac{P_{mis}}{R_{mis}^\alpha}}{\frac{P_{mis}}{c\beta R_{mis}^\alpha}+\frac{(1-1/c)P_{mis}}{\beta R_{mis}^\alpha}} \geq \beta \tag{22}$$

Let P_{clear} be the probability that v can generate a clearance at time t. Based on the above analysis, $P_{clear} \geq (1-1/\eta)P_{no}\cdot P_{one}$. By C we denote a minimum cover of $E_i^{R_M+0.5R_{mis}}$ by some discs D_j. Since we can still assume that Properties 4 and 5 hold before time t_1, and Property 3 is also true before t_1, then the total sending probabilities of nodes in any disc $D_j \in C$ at time t is at most $3\cdot 2^{-\omega}+\phi$ by Lemma 10. Then P_{no} and P_{one} can be bounded as follows:

$$\begin{aligned}
P_{no} &= \prod_{u\in E_i^{R_M+0.5R_{mis}}\setminus D_i} (1-p_u) \\
&\geq \prod_{u\in E_i^{R_M+0.5R_{mis}}} (1-p_u) \\
&\geq \prod_{D_j\in C}\prod_{u\in D_j}(1-p_u) \\
&\geq \prod_{D_j\in C}\left(\frac{1}{4}\right)^{\sum_{u\in D_j} p_u} \\
&\geq \left(\frac{1}{4}\right)^{(3\cdot 2^{-\omega}+\phi)\cdot\chi(R_M+0.5R_{mis},0.5R_{mis})}
\end{aligned} \tag{23}$$

$$\begin{aligned}
P_{one} &= \sum_{v\in\mathcal{A}_i}\left(q_v\cdot\prod_{u\in D_i\setminus\{v\}}(1-p_u)\right) \\
&\geq \sum_{v\in\mathcal{A}_i}\left(q_v\cdot\prod_{u\in D_i}(1-p_u)\right) \\
&\geq \sum_{v\in\mathcal{A}_i} q_v\left(\frac{1}{4}\right)^{\sum_{u\in D_i} p_u} \\
&\geq \sum_{v\in\mathcal{A}_i} q_v\left(\frac{1}{4}\right)^{3\cdot 2^{-\omega+\phi}} \\
&\geq 2^{-\omega-2}\left(\frac{1}{4}\right)^{3\cdot 2^{-\omega+\phi}}
\end{aligned} \tag{24}$$

Thus the probability of $t \in \mathcal{I}$ being a clearance is at least

$$P_{clear} \geq \left(1-\frac{1}{\eta}\right)\cdot\left(\frac{1}{4}\right)^{(3\cdot 2^{-\omega}+\phi)\cdot\chi(R_M+0.5R_{mis},0.5R_{mis})}\cdot 2^{-\omega-2}\left(\frac{1}{4}\right)^{3\cdot 2^{-\omega+\phi}}$$
$$\geq \left(1-\frac{1}{\eta}\right)\cdot 2^{-\omega-2}\cdot\left(\frac{1}{4}\right)^{(3\cdot 2^{-\omega}+\phi)\cdot(\chi(R_M+0.5R_{mis},0.5R_{mis})+1)} \tag{25}$$

Then the probability that none of the $\lambda\log n$ time-slots $t \in \mathcal{I}$ is a clearance is at most $(1-(1-\frac{1}{\eta})\cdot 2^{-\omega-2}\cdot(\frac{1}{4})^{(3\cdot 2^{-\omega}+\phi)\cdot(\chi(R_M+0.5R_{mis},0.5R_{mis})+1)})^{\lambda\log n} \in O(n^{-5})$. Note that the reason for defining $\omega = 6.4$ is that this value maximizes P_{clear}.

Finally, each clearance means that a node in D_i joins state $\mathcal{B}$. For each node w, by Lemma 13, after w joins state $\mathcal{B}$, there will be a node in $E_w^{2R_{mis}}$ joining state $\mathcal{M}$. Since there can be at most constant number of nodes in $E_w^{2R_{mis}}$ that may join state $\mathcal{M}$, w joins state $\mathcal{B}$ for at most constant times and then joins state $\mathcal{M}$ or $\mathcal{S}$. Then there are at most $O(n)$ clearances in D_i. The probability that in the first $O(n)$ such intervals $\mathcal{I}$ in D_i, there is at least one without a clearance is at most $O(n^{-4})$. Thus with probability at least $1-O(n^{-4})$, there is not a violating time-slot t_1 in D_i. Since the same argument is also adaptive for other discs D_j, the claim holds for all discs with probability $1-O(n^{-3})$. □

Claim. *Assume that Property 4 is the first property to be violated and let t_2 be the first time-slot in which the violation occurs. The probability that there exists such a time-slot t_2 during the execution of the algorithm is at most $P_{fail} \in O(n^{-3})$.*

Proof. For a disc D_i, assume that T_c is an interval either between *(i)* two subsequent clearances, or *(ii)* between a clearance and time-slot t_2, or *(iii)* between a clearance and the end of the algorithm, depending on which comes first. Furthermore, let t_c denote the time in which the clearance initiating T_c occurs. Next we show that the probability that Property 4 is violated in this interval is at most $1-O(n^{-5})$. By Algorithm 2, after a clearance, no node $v \in D_i$ is in state $\mathcal{A}$ for the next $4\theta\delta\log^2 n + 2\mu_2\log n$ time-slots. So there is no node entering state $\mathcal{B}$ in D_i in the interval $[t_c,\ldots,t_c+4\theta\delta\log^2 n+2\mu_2\log n]$. Therefore we only need to consider the interval $\mathcal{I} = [t_c + 4\theta\delta\log^2 n + 2\mu_2\log n,\ldots,t_q]$, where t_q is the time-slot of *(i)* the subsequent clearance, *(ii)* time-slot t_2, *(iii)* the end of the algorithm.

Let a *failure* be a time-slot in which a new node joins state $\mathcal{B}$ in D_i without occurring a clearance. In the following, we show that there are no more than $\frac{1}{6e}\kappa\log n$ failures in D_i in the interval $[t_c + 4\theta\delta\log^2 n + 2\mu_2\log n,\ldots,t_q]$ with probability $1-n^{-5}$.

By showing that before $\frac{1}{6e}\kappa\log n$ failures occur, there is at least one clearance with high probability, the claim can be proved since t_q takes place before or at the time of such a clearance.

First we define some random events. $\mathcal{E}_b(t)$ denotes the event of a clearance occurring in D_i at time-slot t and $\mathcal{E}_0(t)$ denotes the event of no node in $\mathcal{A}_i$ sending in time-slot t. Clearly, $\mathcal{E}_b(t)$ is true only if $\mathcal{E}_0(t)$ is false. In the following, we will find a bound on the probability $Pr[\mathcal{E}_b(t)|\overline{\mathcal{E}_0(t)}]$. From the analysis for Inequality (22), we know that if only one node in D_i sends and no node in $E_i^{R_M+0.5R_{mis}} \setminus D_i$ sends, there will be a clearance with probability at least $(1-1/\eta)$. Hence $Pr[\mathcal{E}_b(t)|\overline{\mathcal{E}_0(t)}] \geq (1-1/\eta)\cdot Pr[\mathcal{E}_e(t)]\cdot Pr[\mathcal{E}_1(t)|\overline{\mathcal{E}_0(t)}]$, where $\mathcal{E}_1(t)$ and $\mathcal{E}_e(t)$ denote the event that only one node in D_i sends and the event that no nodes in $E_i^{R_M+0.5R_{mis}} \setminus D_i$ send, respectively. Additionally, denote $\mathcal{E}_+(t)$ as the event that at least two nodes in $\mathcal{A}_i$ send. Then

$$Pr\big[\mathcal{E}_b(t)|\overline{\mathcal{E}_0(t)}\big] \geq (1-1/\eta)\cdot Pr\big[\mathcal{E}_e(t)\big]\cdot\big(1-Pr\big[\overline{\mathcal{E}_1(t)}|\overline{\mathcal{E}_0(t)}\big]\big)$$
$$= (1-1/\eta)\cdot Pr\big[\mathcal{E}_e(t)\big]\cdot\big(1-Pr\big[\mathcal{E}_+(t)|\overline{\mathcal{E}_0(t)}\big]\big)$$
$$= (1-1/\eta)\cdot Pr\big[\mathcal{E}_e(t)\big]\cdot\left(1-\frac{Pr[\mathcal{E}_+(t)]}{Pr[\overline{\mathcal{E}_0(t)}]}\right) \tag{26}$$

The last equation is due to the fact $Pr[\overline{\mathcal{E}_0(t)}|\mathcal{E}_+(t)] = 1$. By the definition of t_q (which is t_2 or earlier), we can still assume that Properties 3, 4, 5 hold. Thus we can use some results established under the assumption of these three properties in the previous proofs. By Inequality (23), we obtained a bound for $Pr[\mathcal{E}_e(t)]$ as follows,

$$Pr\big[\mathcal{E}_e(t)\big] \geq \left(\frac{1}{4}\right)^{(3\cdot 2^{-\omega}+\phi)\cdot\chi(R_M+0.5R_{mis},0.5R_{mis})} \tag{27}$$

We can obtain the following lower bound for $Pr[\overline{\mathcal{E}_0(t)}]$.

$$Pr\big[\overline{\mathcal{E}_0(t)}\big] = 1-\prod_{v\in\mathcal{A}_i}(1-q_v) \geq 1-(1/e)^{\sum_{v\in\mathcal{A}_i} q_v}. \tag{28}$$

Finally, we consider $Pr[\mathcal{E}_+(t)]$,

$$\begin{aligned}
Pr\big[\mathcal{E}_+(t)\big] &= Pr\big[\overline{\mathcal{E}_0(t)}\big] - Pr\big[\mathcal{E}_1(t)\big] \\
&\le 1 - \prod_{v\in\mathcal{A}_i}(1-q_v) - \sum_{v\in\mathcal{A}_i}\Big(q_v\cdot\prod_{u\in\mathcal{A}_i\setminus\{v\}}(1-q_v)\Big) \\
&\le 1 - \Big(\frac{1}{4}\Big)^{\sum_{v\in\mathcal{A}_i} q_v} - \sum_{v\in\mathcal{A}_i}\Big(q_v\cdot\Big(\frac{1}{4}\Big)^{\sum_{v\in\mathcal{A}_i} q_v}\Big) \\
&= 1 - \Big(1+\sum_{v\in\mathcal{A}_i} q_v\Big)\Big(\frac{1}{4}\Big)^{\sum_{v\in\mathcal{A}_i} q_v}. \qquad (29)
\end{aligned}$$

Putting everything together, the probability that there is a clearance if a node in D_i enters $\mathcal{B}$ is at least

$$\begin{aligned}
Pr\big[\mathcal{E}_b(t)|\overline{\mathcal{E}_0(t)}\big] &\ge (1-1/\eta)\cdot\Big(\frac{1}{4}\Big)^{(3\cdot2^{-\omega}+\phi)\cdot\chi(R_M+0.5R_{mis},0.5R_{mis})} \\
&\quad\cdot\Big(1-\frac{1-(1+\sum_{v\in\mathcal{A}_i} q_v)(\frac{1}{4})^{\sum_{v\in\mathcal{A}_i} q_v}}{1-(1/e)^{\sum_{v\in\mathcal{A}_i} q_v}}\Big) \qquad (30)
\end{aligned}$$

Since Property 3 still holds, $\sum_{v\in\mathcal{A}_i} p_v \le 2^{-\omega}$. Additionally, the function in the right hand of the above inequality is minimized at the point $2^{-\omega}$ for $\sum_{v\in\mathcal{A}_i} q_v$ in the range $[0, 2^{-\omega}]$. Hence we have

$$Pr\big[\mathcal{E}_b(t)\,\big|\,\overline{\mathcal{E}_0(t)}\big] \ge 0.23\cdot(1-1/\eta)\cdot\Big(\frac{1}{4}\Big)^{(3\cdot2^{-\omega}+\phi)\cdot\chi(R_M+0.5R_{mis},0.5R_{mis})} \qquad (31)$$

Finally, the probability P_f that there are more than $\frac{1}{6e}\kappa\log n$ failures in D_i in the interval $[t_c+4\theta\delta\log^2 n+2\mu_2\log n,\dots,t_q]$ is

$$P_f \le \Big(1-Pr\big[\mathcal{E}_b(t)\,\big|\,\overline{\mathcal{E}_0(t)}\big]\Big)^{\frac{1}{6e}\kappa\log n} \le n^{-5} \qquad (32)$$

by the definition of κ. Thus, with probability $1-O(n^{-5})$, if as many as $\frac{1}{6e}\kappa\log n$ failures had occurred before t_q, there would have been another clearance before t_q, which contradicts the definition of t_q.

Next we claim that there are only a constant number of new nodes joining $\mathcal{B}$ in D_i per failure time-slot. Let $B(t)$ be the number of nodes in $\mathcal{A}_i$ sending at time t and denote $\mathcal{E}_f(t)$ as the event of a failure. Since if there are two nodes in $\mathcal{A}_i$ sending at t, $\mathcal{E}_f(t)$ must be true, and each node in $\mathcal{A}_i$ decides to send independently, the conditional expectation of $B(t)$ given a failure is $E[B(t)|\mathcal{E}_f(t)] \le E[B(t)]+2 \le \sum_{v\in\mathcal{A}_i} q_v + 2$, which is at most $2^{-\omega}+2$ under the assumption that Property 3 still holds.

Let $n_{\mathcal{I}}$ be the number of nodes joining $\mathcal{B}$ during the interval $\mathcal{I}=[t_c+4\theta\delta\log^2 n+2\mu_2\log n,\dots,t_q]$. By Lemma 14, all nodes that have joined $\mathcal{B}$ by the time t_c would have joined state $\mathcal{M}$ or $\mathcal{S}$ by the time $t_c+4\theta\delta\log^2 n+2\mu_2\log n$. Furthermore, by Algorithm 2 and the definition of clearance, there is no node in D_i joining $\mathcal{B}$ during the interval $[t_c,\dots,t_c+4\theta\delta\log^2 n+2\mu_2\log n]$. So the lemma can be proved by only bounding $n_{\mathcal{I}}$.

Let f denote the number of failures in $\mathcal{I}$. Define a random variable X_j^v for each node $v\in\mathcal{A}_i$ and $j=1,\dots,f$. X_j^v has value 1 if v sends (and enters state $\mathcal{B}$) in the jth failure and 0 otherwise. Then define $X=\sum_{j=1}^{f}\sum_{v\in A_i(t_j)} X_j^v$. Thus X is an upper bound for $n_{\mathcal{I}}$. Note that X_j^v cannot be seen as independent Bernoulli trials because all X_j^v for a node v are not independent. More precisely, $X_j^v=1 \Rightarrow X_{j'}^v=0$ for all $j'>j$ since after sending at time t_j, v joins $\mathcal{B}$. These dependencies make it an upper bound for X if we take X_j^v as independent Bernoulli trials. In the following we compute the upper bound of $n_{\mathcal{I}}$ under this assumption. Then the expectation $E[X]$ is

$$\begin{aligned}
E[X] &= \sum_{j=1}^{f} E\Big[\sum_{v\in A_i(t_j)} X_j^v\Big] \\
&= \sum_{j=1}^{f} E\big[B(t_j)\,\big|\,\mathcal{E}_f(t_j)\big] \\
&\le \big(2^{-\omega}+2\big)\frac{1}{6e}\kappa\log n \\
&< \frac{1}{2e+1}\kappa\log n. \qquad (33)
\end{aligned}$$

The first inequality is assuming there are at most $\frac{1}{6e}\kappa \log n$ failures. Then by the Chernoff bound, the probability that X is larger than $\kappa \log n$ is at most

$$P[X > \kappa \log n] \leq \left(\frac{e^{2e}}{(1+2e)^{1+2e}}\right)^{\frac{\kappa \log n}{2e+1}} \in O\left(n^{-5}\right). \quad (34)$$

As discussed before, assuming X_j^v to be randomly and independently distributed Bernoulli variables yields an upper bound for $n_{\mathcal{I}}$. Thus $P[n_{\mathcal{I}} > \kappa \log n] \leq P[X > \kappa \log n] \in O(n^{-5})$. That is if there are at most $\frac{1}{6e}\kappa \log n$ failures, then at most $\kappa \log n$ nodes join state $\mathcal{B}$ during the interval $\mathcal{I}$ with probability at least $1 - O(n^{-5})$. From above, this assumption is true with probability $1 - O(n^{-5})$. Thus, the probability that in an arbitrary interval T_C after a clearance, Property 4 is not violated before the next clearance is at least $1 - 2O(n^{-5}) \in 1 - O(n^{-5})$.

In each D_i, there are at most $O(n)$ clearances during the execution of the algorithm as shown in the above Claim. So the probability that there are more than $\kappa \log n$ nodes in state $\mathcal{B}_i$ during the execution of the algorithm is at most $1 - O(n^{-4})$. That is, with probability $1 - O(n^{-4})$, Property 4 is not the first violated property. Consider all discs, Property 4 is not the first property to be violated with probability $1 - O(n^{-3})$. □

Claim. *Assume that Property 5 is the first property to be violated and let t_3 be the first time-slot in which the violation occurs. The probability that there exists such a time-slot t_3 during the execution of the algorithm is at most $P_{fail} \in O(n^{-3})$.*

Proof. We prove the claim by showing that when any node joins state $\mathcal{M}$, the *count* values of all its neighbors in state $\mathcal{B}$ are at most $\delta \log^2 n - 2\mu_2 \log n$ with probability at least $1 - O(n^{-2})$. Then after a node joins state $\mathcal{M}$, there will be no neighboring nodes join state $\mathcal{M}$ in the subsequent $2\mu_2 \log n$ timeslots. By the algorithm, all nodes in state $\mathcal{M}$ in any timeslot constitute an independent set, since each node will stay in state $\mathcal{M}$ for $2\mu_2 \log n$ timeslots.

Assume that v is the node that violates Property 5 at time t_3. Let u be a neighbor of v which joins state $\mathcal{M}$ at time $t_u \leq t_3$. Since v violates Property 5, $t_3 - t_u < 2\mu_2 \log n$. Thus at time t_u, v' *count* is larger than $\delta \log^2 n - 2\mu_2 \log n$. By the algorithm, u has started increasing its *count* since the time $t_u - \delta \log^2 n$ and will not reset the *count*'s value. Every node (e.g., v) in u's neighborhood with *count* larger than $\delta \log^2 n - 2\mu_2 \log n$ at time t_u has started increasing its *count* since the time $t_u - \delta \log^2 n + 2\mu_2 \log n$. Because v is the first node which violates Property 5 and Property 5 is the first violated property, we can still assume that all three properties are still correct before t_3. Using a similar analysis in the proof of Lemma 12, we can prove that u will successfully transmit the message $m_B(count_u)$ to all its neighbors in $\frac{\delta}{2} \log^2 n$ timeslots with probability at least $1 - O(n^{-4})$. Thus u can successfully send a message $m_B(count_u)$ to all its neighbors during the interval $[t_u - \delta \log^2 n + 2\mu_2 \log n, t_u - 1]$. Then by Algorithm 2, all its neighbors in state $\mathcal{B}$, including v, reset their *count* values to be away from m' *count* by at least $2\mu_2 \log n$. Then v is impossible to join state $\mathcal{M}$ at time t_3. In other words, when u joins state $\mathcal{M}$, there will be no such a timeslot t_3 with probability at least $1 - O(n^{-4})$. Since each node only joins state $\mathcal{M}$ for at most once, the claim is correct during the execution of the algorithm with probability at least $1 - O(n^{-3})$. □

Finally, combining all above, we give the proof of Lemma 2.

Proof of Lemma 2. We first bound the time v spends in executing the algorithm. If v does not receive a message m_A or *DoNotTransmit* from a neighbor node for $4\theta\delta \log^2 n + 2\mu_2 \log n$ time-slots after entering state $\mathcal{W}$, it will join state $\mathcal{A}$. Then unless it receives a message m_A or *DoNotTransmit*, its sending probability will increase to $2^{-\omega-2}$ after $(\log n - 1)\lambda \log n$ time-slots. If v still does not receive m_A or *DoNotTransmit* in the subsequent $\lambda \log n$ time-slots, the probability that v does not send during these $\lambda \log n$ time-slots is at most $(1 - 2^{-\omega-2})^{\lambda \log n} \in O(n^{-4})$. Thus after v starts the algorithm for at most $4\theta\delta \log^2 n + \lambda \log^2 n + 2\mu_2 \log n$ timeslots, there will be three cases for v: *(i)* v joins state $\mathcal{B}$; *(ii)* v receives a message m_A from a neighbor and restarts the algorithm; *(iii)* v receives a message *DoNotTransmit* from a neighbor. For case *(i)*, by Lemma 14 and Algorithm 2, with probability at least $1 - O(n^{-4})$, there will be one node in T_v joins state $\mathcal{M}$ and quits the MIS algorithm after at most $4\theta\delta \log^2 n + 2\mu_2 \log n$ timeslots. For case *(ii)*, it means that one of v's neighbors joins state $\mathcal{B}$. Similarly, by Lemma 14 and Algorithm 2, there will be a node in $E_v^{2R_{mis}}$ joins state $\mathcal{M}$ and quits the algorithm after at most $4\theta\delta \log^2 n + 2\mu_2 \log n$ timeslots with probability $1 - O(n^{-4})$. For case *(iii)*, it means that one of v's neighbors joins state $\mathcal{M}$. Combining the above analysis, after v starts or restarts the algorithm for $8\theta\delta \log^2 n + \lambda \log^2 n + 4\mu_2 \log n$ timeslots, there will be a node in $E_v^{2R_{mis}}$ joins state $\mathcal{M}$ and quits the MIS algorithm with probability at least $1 - O(n^{-4})$. Using a similar argument as in Lemma 14, after at most $\theta(8\theta\delta \log^2 n + \lambda \log^2 n + 4\mu_2 \log n)$ timeslots, there will be a node in T_v joining state $\mathcal{M}$. From then, by Lemma 12, if v is not the node, it will be forced joining state $\mathcal{S}$ in $O(\log n)$ timeslots with probability at least $1 - O(n^{-4})$. Thus, with probability at least $1 - O(n^{-4})$, it takes at most $O(\log^2 n)$ timeslots in executing Algorithm MIS.

Furthermore, as stated in Lemma 12, with probability $1 - O(n^{-4})$, a node v in state $\mathcal{M}$ can send the *DoNotTransmit* message to all nodes in T_v before any of them joining state $\mathcal{M}$, which means that with probability $1 - O(n^{-4})$, there is not any node in $N(v)$ staying in state $\mathcal{M}$ when v is in state $\mathcal{M}$. By Algorithm 1, each node joins state $\mathcal{M}$ for at most once. Hence, with probability at least $1 - O(n^{-3})$, in any timeslot t, the independent set computed by Algorithm MIS is correct.

Note that all above analysis is based on Properties 3, 4 and 5. By Lemma 15, these three properties are correct with probability $1 - O(n^{-3})$. Then we prove the lemma. □

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